Approximation by functions of compact support on irregular domains *

António M. Caetano †

17th December 2004

Abstract

General Besov and Triebel-Lizorkin spaces on domains with irregular boundary are compared with the completion, in those spaces, of the subset of infinitely continuously differentiable functions with compact support in the same domains. It turns out that the set of parameters for which those spaces coincide is strongly related with the fractal dimension of the boundary of the domains.

Key words: atom, ball, Besov, boundary, compact, condition, density, dimension, domain, d-set, fractal, Hausdorff, interior, irregular, Minkowski, regular, space, support, trace, Triebel, Triebel-Lizorkin.

The results described here were motivated by the following one proved in [8]:
"Let Ω be a bounded C^∞-domain in $\mathbb{R}^n$. Then

$$F^s_{pq}(\Omega) = \overset{\circ}{F}{}^s_{pq}(\Omega) \tag{1}$$

if, and only if, one of the following two conditions holds:

(i) $-\infty < s < 1/p$, $0 < p, q < \infty$;

(ii) $s = 1/p$, $1 < p < \infty$, $0 < q < \infty$."

The "if" part of the proof was made via the technique of atomic decomposition and a close inspection made it clear that something could be done for domains with irregular boundary (actually, such a possibility was independently noticed in [3], though in a narrower context than the one we are going to consider here). The

*1991 *Mathematics Subject Classification*: 46E35.

†Research partially supported by the Deutscher Akademisher Austauschdienst and the Fundação para a Ciência e a Tecnologia.

critical value for s should then, instead of $1/p$, be $(n-D)/p$, with D the upper Minkowski dimension of $\partial\Omega$. Note that in the case above, where $\partial\Omega$ is C^∞, D is $n-1$, so that $(n-D)/p$ equals $1/p$, as it should be.

Here we deal not only with $F^s_{pq}(\Omega)$ but also with $B^s_{pq}(\Omega)$, where we consider the definition by restriction from the corresponding spaces defined on $\mathbb{R}^n$ (for details about this, refer to [6]). $\overset{\circ}{F}{}^s_{pq}(\Omega)$ and $\overset{\circ}{B}{}^s_{pq}(\Omega)$ stand for the completion of $C_0^\infty(\Omega)$ in $F^s_{pq}(\Omega)$ and $B^s_{pq}(\Omega)$, respectively. Ω has always the minimum requirement of being an open, non-empty set in some fixed $\mathbb{R}^n$ and we call it a domain. The minimum requirements for the parameters s, p, q which are assumed here are $s \in \mathbb{R}$ and $0 < p, q < \infty$ (so the case $p = \infty \vee q = \infty$ is ruled out from the very beginning). When there are some restrictions to be made to this general setting (both for Ω and the parameters), we shall only mention such restrictions.

As pointed out above, we shall also need to deal with upper Minkowski dimensions (u.M.d., for short) D of bounded non-empty sets $\Gamma \in \mathbb{R}^n$, but only to the extent of the well-known consequence that, for such a Γ, $c_1 r^{-d}$ balls of radius $c_2 r$ are enough to cover Γ, for r small and c_1, c_2 positive constants, where d can be any number $> D$. In any case, we are not, in general, going to present proofs here (for proofs, please check [2]).

Proposition 1 *Let Ω be a bounded domain such that $\partial\Omega$ has u.M.d. D, for some $D < n$. Then, for any $A \in \{B, F\}$,*

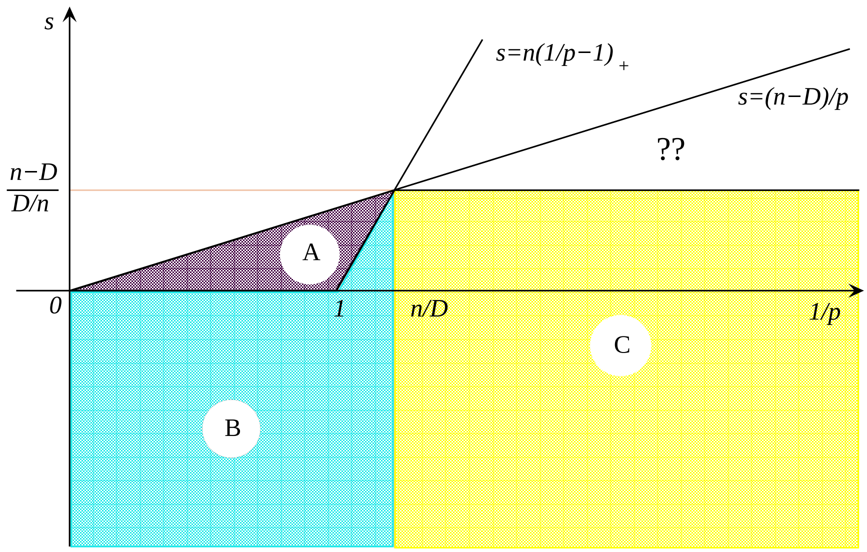
$$A^s_{pq}(\Omega) = \overset{\circ}{A}{}^s_{pq}(\Omega) \tag{2}$$

if either

(i) $p > \frac{D}{n}$ *and* $s < \frac{n-D}{p}$ *or*

(ii) $p \le \frac{D}{n}$ *and* $s < \frac{n-D}{D/n}$.

The proof is made first for spaces indexed in zone A of the $(1/p, s)$-diagram shown in the picture and it is enough to deal with B^s_{pq}-spaces. Basically what is proved is that, under the assumptions made, any $f \in \mathcal{S}$ can be arbitrarily approximated in $B^s_{pq}(\mathbb{R}^n)$ by $C_0^\infty(\Omega)$-functions with support in $(\partial\Omega)^c$ (and it is here that the atomic technique is used), as this easily implies our conclusion. The reason for restricting, in a first phase, the proof to spaces indexed in zone A of the picture is that below the line $s = n(1/p - 1)_+$ the proof would require what are called moment conditions, and it is not clear how to guarantee that in the general case, or if that is at all possible. The result is extended to zones B (which completes the proof of case (i)) and C (which corresponds to case (ii)) by means of well-known embedding operators and the following simple but quite useful result:



Lemma 2 *Assume that the equality* $A^{s_1}_{p_1q_1}(\Omega) = \overset{\circ}{A}{}^{s_1}_{p_1q_1}(\Omega)$ *holds for some* $s_1 \in \mathbb{R}$, $0 < p_1, q_1 < \infty$ *and* $A \in \{B, F\}$. *If for some* $G \in \{B, F\}$, $s_2 \in \mathbb{R}$, $0 < p_2, q_2 < \infty$ *the continuous embedding* $A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow G^{s_2}_{p_2q_2}(\Omega)$ *is true, then the equality* $G^{s_2}_{p_2q_2}(\Omega) = \overset{\circ}{G}{}^{s_2}_{p_2q_2}(\Omega)$ *also holds.*

In order to deal with zone ?? of the picture, we need the so-called ball condition ([6], with modifications):

Definition 3 *A non-empty set* $\Gamma \subset \mathbb{R}^n$ *is said to satisfy the ball condition if*

$$\exists_{\eta\in]0,1[}\,\forall_{x\in\Gamma}\,\forall_{r\in]0,1[}\,\exists_{y\in\mathbb{R}^n}\ B(y,\eta r) \subset B(x,r)\ \wedge\ B(x,\eta r)\cap\overline{\Gamma} = \emptyset,$$

where the notation $B(z,s)$ *means the closed ball centred at* z *with radius* s.

Proposition 4 *Let* Ω *be a bounded domain such that* $\partial\Omega$ *has u.M.d.* D *and satisfies the ball condition. Then*

$$s < \frac{n-D}{p} \Rightarrow A^s_{pq}(\Omega) = \overset{\circ}{A}{}^s_{pq}(\Omega) \text{ for any } A \in \{B, F\}.$$

The proof is a modification of the preceding one: moment conditions have to be created and this is done by means of a tricky construction (which can be seen in a similar situation in [9]) possible in the case the ball condition is at our disposal.

There is a nice corollary for which we need the notion of d-set:

Definition 5 *Let* Γ *be a non-empty closed subset of* $\mathbb{R}^n$ *and* $d \in]0,n]$. Γ *is said to be a* d*-set if*

$$\exists c_1, c_2 > 0 : \forall\gamma\in\Gamma, \forall r \in]0,1],\ c_1 r^d \le \mathcal{H}^d(B(\gamma,r)\cap\Gamma) \le c_2 r^d,$$

where $\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure on $\mathbb{R}^n$ and $B(\gamma, r)$ stands for the closed ball centred at γ and with radius r.

It is known that for d-sets the Minkowski dimension equals the Hausdorff dimension, which is d. Moreover, any d-set with $d < n$ satisfies the ball condition (cf. [1]). Therefore

Corollary 6 *Let Ω be a bounded domain such that $\partial\Omega$ is a d-set, for some $d < n$. Then*

$$s < \frac{n-d}{p} \Rightarrow A^s_{pq}(\Omega) = \overset{\circ}{A}{}^s_{pq}(\Omega) \text{ for any } A \in \{B, F\}.$$

Our results in the converse direction depend on trace results (not always available in the literature) on d-sets.

We take the notion of trace $\mathrm{tr}_\Gamma f$ for functions f of $A^s_{pq}(\mathbb{R}^n)$ ($A \in \{B, F\}$) on d-sets Γ as in [7, pp. 138-139]: $\mathrm{tr}_\Gamma \varphi = \varphi|_\Gamma$, that is, is defined pointwise, when $\varphi \in \mathcal{S}$; tr_Γ is defined by completion for all remaining functions of $A^s_{pq}(\mathbb{R}^n)$ whenever it is possible to find $c > 0$ such that

$$\|\mathrm{tr}_\Gamma \varphi | L_p(\Gamma)\| \leq c\, \|\varphi | A^s_{pq}(\mathbb{R}^n)\|, \ \varphi \in \mathcal{S}$$

($L_p(\Gamma)$ considered with respect to the measure $\mathcal{H}^d|_\Gamma$). In particular, we are not claiming that the trace always exists! When it does — that is, when the above approach succeeds — we shall say that "$\mathrm{tr}_\Gamma A^s_{pq}(\mathbb{R}^n)$ exists". Note that, by definition, with this phrase we are also saying that the trace operator

$$\mathrm{tr}_\Gamma : A^s_{pq}(\mathbb{R}^n) \longrightarrow L_p(\Gamma)$$

is linear and bounded.

We take the following as an easy consequence of results in [7]:

Lemma 7 *If Γ is a d-set, $d < n$, then*

$$s > \frac{n-d}{p} \Rightarrow \mathrm{tr}_\Gamma A^s_{pq}(\mathbb{R}^n) \text{ exists, } A \in \{B, F\}.$$

We also need to deal with the question of existence of a trace $\mathrm{Tr}_{\partial\Omega} f$ on $\partial\Omega$ for functions f of $A^s_{pq}(\Omega)$, at least for some spaces $A^s_{pq}(\Omega)$.

Recall that $H^s_p(\mathbb{R}^n) \equiv F^s_{p2}(\mathbb{R}^n)$, $s > 0$, $p > 1$, are spaces of Bessel potentials and define $H^s_p(\Omega)$, $s > 0$, $p > 1$, as being equal to $F^s_{p2}(\Omega)$ — so, it is defined by restriction.

The following notion is taken from [9], with modifications:

Definition 8 *A domain Ω is said to be interior regular if*

$$\exists_{c>0}\,\forall_{x\in\partial\Omega}\,\forall_{cube\ Q\ \ centred\ at\ x\ \ with\ side\ length\ \leq 1}\ |\Omega\cap Q|\geq c\,|Q|.$$

Proposition 9 *Let Ω be an interior regular domain such that $\partial\Omega$ is a d-set. If $p>1$ and $s>(n-d)/p$ then*

$$f\in H^s_p(\Omega)\Rightarrow \textit{all } u\in H^s_p(\mathbb{R}^n) \textit{ such that } u|_\Omega=f \textit{ have the same trace on } \partial\Omega.$$

The proof of this result takes advantage of the fact $H^s_p(\mathbb{R}^n)$ is a space of Bessel potentials, of continuity properties of potentials as described in [5] and of the equivalent definition of trace given, for any $u\in H^s_p(\mathbb{R}^n)$ (and under the hypothesis of the proposition), by the restriction to $\partial\Omega$ of

$$x\mapsto \lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)}u(y)dy. \tag{3}$$

The method of proof also gives for the common trace to $\partial\Omega$ of any $u\in H^s_p(\mathbb{R}^n)$ such that $u|_\Omega=f$ the representative

$$x\mapsto \lim_{r\to 0}\frac{1}{|B(x,r)\cap\Omega|}\int_{B(x,r)\cap\Omega}f(x)dx,\ x\in\partial\Omega, \tag{4}$$

and is modelled on proofs of related results in [4] or [10].

This proposition makes it clear that (under the assumptions made) the trace of any $f\in H^s_p(\Omega)$ to $\partial\Omega$ can be defined (and how it can be defined). It is also immediate that such a notion of trace gives rise to an operator

$$\mathrm{Tr}_{\partial\Omega}: H^s_p(\Omega)\to L_p(\partial\Omega)$$

which is linear and bounded.

Now we are ready for the following converse to Corollary 6.

Proposition 10 *Let Ω be an interior regular domain such that $\partial\Omega$ is a d-set. If $p>1$, then*

$$s>\frac{n-d}{p}\Rightarrow A^s_{pq}(\Omega)\neq \overset{\circ}{A}{}^s_{pq}(\Omega),\ \textit{for any } A\in\{B,F\}.$$

We note that once we have the result for $H^s_p(\Omega)$ the rest is a matter of using well-known embeddings. For $H^s_p(\Omega)$ we argue in the following way:

On one hand, there are elements of $H^s_p(\Omega)$ with a non-zero trace on $\partial\Omega$: for example, any $\varphi|_\Omega$ such that $\varphi\in C^\infty_0(\mathbb{R}^n)$ and $\varphi\equiv 1$ on $B(\gamma,1)$, for some $\gamma\in\partial\Omega$. On the other hand, if $H^s_p(\Omega)=\overset{\circ}{H}{}^s_p(\Omega)$ were true, any $f\in H^s_p(\Omega)$ could be arbitrarily approximated in $H_p^s(\Omega)$ by functions in $C_0^\infty(\Omega)$, the trace of which is zero. Hence, the continuity of the trace operator would lead to $\mathrm{Tr}_{\partial\Omega} f = 0$ in $L_p(\partial\Omega)$, which contradicts what we said before.

By using embedding operators and Lemma 2, the above result can also be extended to some values of $p \leq 1$, namely those for which $s > n/p - d$ (note this implies $s > (n-d)/p$ automatically).

We have no general result with unrestricted p. However, the following classes of examples show that the problem should be further investigated.

Proposition 11 *Let ω be a bounded domain in $\mathbb{R}^n$ and Γ a d-set, $d < n$, such that $\Gamma \subset \omega$. Let $\Omega \equiv \omega \setminus \Gamma$. If $s > n(1/p - 1)_+$, then*

$$s > \frac{n-d}{p} \Rightarrow A_{pq}^s(\Omega) \neq \overset{\circ}{A}{}_{pq}^s(\Omega) \text{ for any } A \in \{B, F\}.$$

Proposition 12 *For each $D \in [n-1, n[$ (with $n \geq 2$) there is a bounded simply connected domain Ω such that $\partial\Omega$ has u.M.d. equal to D and for which we have*

$$s > \frac{n-D}{p} \Rightarrow A_{pq}^s(\Omega) \neq \overset{\circ}{A}{}_{pq}^s(\Omega) \text{ for any } A \in \{B, F\}.$$

The proofs of both propositions take advantage of trace results for spaces in $\mathbb{R}^n$, as given by Lemma 7.

References

[1] A. M. Caetano, *On fractals wich are not so terrible*, in preparation.

[2] ———, *Approximation by functions of compact support in Besov-Triebel-Lizorkin spaces on irregular domains*, preprint, 1999.

[3] W. Farkas and N. Jacob, *Sobolev spaces on non-smooth domains and Dirichlet forms related to subordinate reflecting diffusions*, preprint, 1999.

[4] A. Jonsson and H. Wallin, *Function spaces on subsets of $\mathbb{R}^n$*, Math. Reports, vol. 2, Harwood Acad. Publ., London, 1984.

[5] N. G. Meyers, *Continuity properties of potentials*, Duke Math. J. **42** (1975), 157–166.

[6] H. Triebel, *Theory of function spaces*, Birkhäuser, Basel Boston Stuttgart, 1983.

[7] ———, *Fractals and spectra*, Birkhäuser, Basel Boston Berlin, 1997.

[8] ———, *Decompositions of function spaces*, Topics in Nonlinear Analysis, the Herbert Amann Aniversary Volume, Birkhäuser, Basel, 1999, pp. 691–730.

[9] H. Triebel and H. Winkelvoss, *Intrinsic atomic characterizations of function spaces on domains*, Math. Z. **221** (1996), 647–673.

[10] H. Wallin, *The trace to the boundary of Sobolev spaces on a snowflake*, Manuscr. Math. **73** (1991), 117–125.

Departamento de Matemática
Universidade de Aveiro
3810 AVEIRO
Portugal
E-mail: acaetano@mat.ua.pt