Approximation by functions of compact support in Besov-Triebel-Lizorkin spaces on irregular domains *

António M. Caetano †

January 2000

Abstract

General Besov and Triebel-Lizorkin spaces on domains with irregular boundary are compared with the completion, in those spaces, of the subset of infinitely continuously differentiable functions with compact support in the same domains. It turns out that the set of parameters for which those spaces coincide is strongly related with the fractal dimension of the boundary of the domains.

1 Introduction

Let $A^s_{pq}(\Omega)$ stand either for the Besov space $B^s_{pq}(\Omega)$ (case $A = B$) or the Triebel-Lizorkin space $F^s_{pq}(\Omega)$ (case $A = F$) on the bounded domain Ω. In this paper we study the possibility of the set $C_0^\infty(\Omega)$ being dense in $A^s_{pq}(\Omega)$ against some measure of the fractality of the boundary $\partial\Omega$ of Ω. It turns out, in section 2, that if d is the upper Minkowski dimension of $\partial\Omega$ and, moreover, $\partial\Omega$ satisfies the ball condition, then we have density as long as $s < (n-d)/p$, $0 < p, q < \infty$ (cf. Proposition 2.5). We also point out that in many situations the ball condition is not really necessary for the density to hold (cf. Proposition 2.2).

*1991 *Mathematics Subject Classification*: 46E35.

†Research partially supported by the Deutscher Akademisher Austauschdienst and the Fundação para a Ciência e a Tecnologia.

After obtaining, as a consequence, the same result when $\partial\Omega$ is a d-set, $d < n$ (cf. Corollary 2.7), we investigate, in section 3, converse results to the above mentioned. These turn out to be closely related to the problem of the existence of a (continuous) trace to the boundary $\partial\Omega$ and we are able to prove that the density does not hold if $s > (n - d)/p$, as long as $\partial\Omega$ is a d-set, $1 \le p < \infty$, $0 < q < \infty$ and Ω is interior regular (cf. Proposition 3.7). Actually, we even prove that the density does not hold for some values of $p < 1$, but here the results are less satisfactory.

In section 4 of the paper we give examples, for each possible $0 < p, q < \infty$, where the density fails for any $s > (n-d)/p$, where d is the upper Minkowski dimension of the boundary of the domain in question ($d < n$). This, of course, is not as strong as having a converse to the results of section 2, but clearly shows that the upper bound $(n - d)/p$ for s cannot in general be improved in the results of that section.

The present work was prompted by corresponding results for F-spaces on smooth domains proved by Triebel in [11] (in which case $d = n - 1$). Actually, we benefited very much from discussions with Prof. Triebel about this and related subjects during three stays in Jena along the academic year 1998/99. We take the opportunity to thank Prof. Triebel, as well as other people in his group, for the stimulating atmosphere and DAAD and FCT for the grants which supported us over that period.

2 Conventions and density results

As mentioned above, $A^s_{pq}(\Omega)$ will always stand either for $B^s_{pq}(\Omega)$ or $F^s_{pq}(\Omega)$, where these are the Besov and Triebel-Lizorkin spaces, the definition of which can, for example, be seen in [9]. Here we just mention that they are defined by restriction from the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$, respectively (in particular, in the cases where $F^s_{pq}(\mathbb{R}^n)$ coincides with a Sobolev space $W^s_p(\mathbb{R}^n)$, $F^s_{pq}(\Omega)$ will not always be the same as the classical Sobolev space $W^s_p(\Omega)$: the irregularity of the boundary $\partial\Omega$ of Ω can make the difference). Moreover, when writing $A^s_{pq}(\Omega)$ we will always assume the minimum requirements that $s \in \mathbb{R}$, $0 < p, q < \infty$ and Ω is a domain (that is, an open, non-empty subset of $\mathbb{R}^n$, for some fixed $n \in \mathbb{N}$). This will hold throughout all the paper and should be borne in mind when reading the assertions of the various results stated, as in each assertion we will only mention the restrictions to this general setting. We also should like to recall that those minimum requirements for s, p and q in particular imply that the Schwartz space $\mathcal{S}$ is dense in $A^s_{pq}(\mathbb{R}^n)$.

By $\overset{\circ}{A}{}^s_{pq}(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ in $A^s_{pq}(\Omega)$, for $A \in \{B, F\}$.

We recall that the upper Minkowski dimension (u.M.d., for short) D of a bounded non-empty $\Gamma \subset \mathbb{R}^n$ is

$$D \equiv \inf \{d \geq 0 : \limsup_{r \to 0+} \frac{|\Gamma_r|}{r^{n-d}} < \infty\},$$

where, in this context, $|\cdot|$ stands for Lebesgue measure in $\mathbb{R}^n$ and

$$\Gamma_r \equiv \{x \in \mathbb{R}^n : \text{dist}\,(x, \Gamma) < r\}. \tag{1}$$

Note also that $D \in [n-1, n]$ when $\Gamma = \partial\Omega$, with Ω a bounded domain.

Lemma 2.1 *Assume that the equality* $A^{s_1}_{p_1 q_1}(\Omega) = \overset{\circ}{A}{}^{s_1}_{p_1 q_1}(\Omega)$ *holds for some* $s_1 \in \mathbb{R}$, $0 < p_1, q_1 < \infty$ *and* $A \in \{B, F\}$. *If for some* $G \in \{B, F\}$, $s_2 \in \mathbb{R}$, $0 < p_2, q_2 < \infty$ *the continuous embedding* $A^{s_1}_{p_1 q_1}(\Omega) \hookrightarrow G^{s_2}_{p_2 q_2}(\Omega)$ *is true, then the equality* $G^{s_2}_{p_2 q_2}(\Omega) = \overset{\circ}{G}{}^{s_2}_{p_2 q_2}(\Omega)$ *also holds.*

Proof. Given any $\varphi \in \mathcal{S}$, $\varphi|_\Omega \in A^{s_1}_{p_1 q_1}(\Omega) = \overset{\circ}{A}{}^{s_1}_{p_1 q_1}(\Omega)$, so that there exists $(\psi_k)_{k \in \mathbf{N}} \subset C_0^\infty(\Omega)$ such that, as $k \to \infty$, $\psi_k \to \varphi|_\Omega$ in $A^{s_1}_{p_1 q_1}(\Omega) \hookrightarrow G^{s_2}_{p_2 q_2}(\Omega)$, so that also $\psi_k \to \varphi|_\Omega$ in $G^{s_2}_{p_2 q_2}(\Omega)$ and therefore $\varphi|_\Omega \in \overset{\circ}{G}{}^{s_2}_{p_2 q_2}(\Omega)$. The result follows by completion. □

Proposition 2.2 *Let* Ω *be a bounded domain such that* $\partial\Omega$ *has u.M.d.* D, *for some* $D < n$. *Then, for any* $A \in \{B, F\}$,

$$A^s_{pq}(\Omega) = \overset{\circ}{A}{}^s_{pq}(\Omega) \tag{2}$$

if either

(i) $p > \frac{D}{n}$ *and* $s < \frac{n-D}{p}$ *or*

(ii) $p \leq \frac{D}{n}$ *and* $s < \frac{n-D}{D/n}$.

Proof. *Step 1.* We first deal with the case $A = B$ and start by showing the following:

"If any $f \in \mathcal{S}$ can be arbitrarily approximated in $B^s_{pq}(\mathbb{R}^n)$ by functions of

$$\mathcal{S}_\Omega \equiv \{g \in \mathcal{S} : g(x) = 0 \text{ for all } x \in \Omega \text{ in some neighbourhood of } \partial\Omega\ \}, \quad (3)$$

then $B^s_{pq}(\Omega) = \overset{\circ}{B}{}^s_{pq}(\Omega)$".

Indeed, given any $\varphi \in \mathcal{S}$, the assumption assures that there exists $(\psi_k)_{k\in\mathbb{N}} \subset \mathcal{S}_\Omega$ such that, as $k \to \infty$, $\psi_k \to \varphi$ in $B^s_{pq}(\mathbb{R}^n)$, so that, by restriction, $\psi_k|_\Omega \to \varphi|_\Omega$ in $B^s_{pq}(\Omega)$, where, clearly, $(\psi_k|_\Omega)_{k\in\mathbb{N}} \subset C^\infty_0(\Omega)$. Therefore $\varphi|_\Omega \in \overset{\circ}{B}{}^s_{pq}(\Omega)$ and the conclusion follows by completion.

Step 2. Consider a family

$$\{\varphi_{jm} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\} \quad (4)$$

of smooth functions in $\mathbb{R}^n$ for which there are constants $c > 0$ and $c_\gamma > 0$, $\gamma \in \mathbb{N}^n_0$, such that

(a) $\operatorname{supp}\varphi_{jm} \subset B_{jm} \equiv \{y \in \mathbb{R}^n : |y - 2^{-j}m| \leq c\,2^{-j}\}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$;

(b) $|D^\gamma\varphi_{jm}(x)| \leq c_\gamma 2^{j|\gamma|}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $\gamma \in \mathbb{N}^n_0$, $x \in \mathbb{R}^n$;

(c) $\sum_{m\in\mathbb{Z}^n} \varphi_{jm}(x) = 1$, $x \in \mathbb{R}^n$, $j \in \mathbb{N}_0$.

Note that, for each $j \in \mathbb{N}_0$, $\{\varphi_{jm} : m \in \mathbb{Z}^n\}$ is a partition of unity in $\mathbb{R}^n$. Such families clearly exist and we just take (and fix) one of them.

It is now possible to prove the following:

"Given any $f \in \mathcal{S}$,

$$s > \sigma_p \ \Rightarrow\ \left\| \sum_m \varphi_{jm} f\,|B^s_{pq}(\mathbb{R}^n) \right\| \leq c\,2^{j(s-(n-d)/p)}, \text{ large } j\text{ ''}, \quad (5)$$

where

$$\sigma_p \equiv n(\frac{1}{p} - 1)_+,$$

$c > 0$ is a constant, d can be arbitrarily fixed such that $d > D$ and the (finite) sum $\sum_m$ is taken, for each j, over all $m \in \mathbb{Z}^n$ such that B_{jm} intersects

$$(\overset{\circ}{\partial\Omega})_{2^{-j}} \equiv (\partial\Omega)_{2^{-j}} \cap \Omega. \tag{6}$$

The idea of proof is that $\sum_m \varphi_{jm} f$ is almost in atomic form $\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}$, with all $\lambda_{\nu m}$ and $a_{\nu m}$ being zero except for $\nu = j$ and m in the range considered above. If we put it in atomic form, we can use the quasi-norm in the sequence space b_{pq} to estimate the above quasi-norm in $B^s_{pq}(\mathbb{R}^n)$ — for the theory involved here, including the definition of b_{pq}, please refer to [10, pp. 70-71]. And, in fact, it is not difficult to see that

$$\sum_m \varphi_{jm} f = \sum_m (c(K,f)2^{j(s-n/p)})(c(K,f)^{-1}2^{-j(s-n/p)}\varphi_{jm} f), \tag{7}$$

where K is fixed not less than $(1+[s])_+$, $c(K,f)$ is a positive constant and $c(K,f)^{-1}2^{-j(s-n/p)}\varphi_{jm} f$ are $(s,p)_{K,-1}$-atoms. Since we are assuming $s > \sigma_p$, no moments conditions are required, so that we have obtained in this way an atomic representation for $\sum_m \varphi_{jm} f$, and we can write, on the basis of Theorem 13.8 of [10, p. 75], that

$$\begin{aligned} \left\| \sum_m \varphi_{jm} f \,|B^s_{pq}(\mathbb{R}^n) \right\| &\leq c_1 (\sum_m |c(K,f)2^{j(s-n/p)}|^p)^{1/p} \\ &\leq c_1 c(K,f) 2^{j(s-n/p)} (\sum_m 1)^{1/p} \\ &\leq c\, 2^{j(s-(n-d)/p)}, \end{aligned} \tag{8}$$

large j, where the last inequality comes from the fact that $\sum_m 1$ is bounded above by positive constant times 2^{jd}, which in turn is a direct consequence of the hypothesis that $\partial\Omega$ has upper Minkowski dimension D (recall also that we are assuming $d > D$).

Step 3. Note now that, if $\sigma_p < s < (n-D)/p$, it is possible to choose $d > D$ such that $s < (n-d)/p$. Using this d in (5) we see we have proved that any $f \in \mathcal{S}$ is the limit, in $B^s_{pq}(\mathbb{R}^n)$, of the sequence $(f - \sum_m \varphi_{jm} f)_{j\in\mathbb{N}_0} \subset \mathcal{S}_\Omega$. By Step 1, we can thus conclude that

$$B^s_{pq}(\Omega) = \overset{\circ}{B}{}^s_{pq}(\Omega) \tag{9}$$

for $\sigma_p < s < (n-D)/p$.

Step 4. Assume that (i) holds.

If $p \geq 1$, then $\sigma_p = 0$ and, by what we have already seen, (9) is true with s_1 instead of s as long as $0 < s_1 < (n-D)/p$. Since we are assuming $D < n$, there really exists such a s_1. On the other hand, for $s \leq 0$ we have $B^{s_1}_{pq}(\Omega) \hookrightarrow B^s_{pq}(\Omega)$ (see, for example, [9, p. 47] for the case $\Omega = \mathbb{R}^n$ and take advantage of the definition by restriction for the case of general bounded Ω). Lemma 2.1 then applies to give us (9) also for such values of s.

If $D/n < p < 1$, then $\sigma_p = n(1/p-1)$ and $D/n < p \Leftrightarrow \sigma_p < (n-D)/p$, so that it is possible to choose s_1 such that $\sigma_p < s_1 < (n-D)/p$, and (9) holds with s_1 instead of s. For $s \leq \sigma_p$ we have again the continuous embedding $B^{s_1}_{pq}(\Omega) \hookrightarrow B^s_{pq}(\Omega)$ and, consequently, also (9), due to Lemma 2.1

Step 5. Assume that (ii) holds.

Since now $p \leq D/n < 1$, we have $\sigma_p = n(1/p-1)$. Apply (i) with $D/n+\varepsilon$ instead of p, for some $\varepsilon > 0$ such that $s < (n-D)/(D/n+\varepsilon)$ (which is clearly possible because of the assumption $s < (n-D)/(D/n)$) in order to obtain (9) with $D/n+\varepsilon$ instead of p. Since $B^s_{D/n+\varepsilon,q}(\Omega) \hookrightarrow B^s_{pq}(\Omega)$ (cf. [9, p. 197] for the case of bounded C^∞-domains; for general bounded domains the result can be reduced to that one by playing a little bit with the definition of the spaces by restriction), Lemma 2.1 again gives what we want, namely (9) for the parameters as in (ii).

Step 6. It only remains to deal with the case $A = F$. Since $B^s_{p,\min\{p,q\}}(\Omega) \hookrightarrow F^s_{pq}(\Omega)$ (see again [9, p. 47] for the case $\Omega = \mathbb{R}^n$) and (9) holds with $\min\{p,q\}$ instead of q either under the hypothesis (i) or under the hypothesis (ii), the conclusion follows by yet another application of Lemma 2.1. □

We remark here that the possibility that (2) holds for Bessel potential spaces under the assumption $s < (n-D)/p$ was independently noticed in [4] under some more restrictive hypotheses on Ω, $\partial\Omega$ and the parameters. In any case, such a result is contained in the proposition above.

We shall see in a moment that, with a convenient extra assumption on $\partial\Omega$, it is possible to guarantee that the density result (2) holds for all $s < (n-D)/p$ even if $p < D/n$.

The following definition is taken from [10, p.142] with modifications:

Definition 2.3 *A non-empty set* $\Gamma \subset \mathbb{R}^n$ *is said to satisfy the ball condition if*

$$\exists_{\eta\in]0,1[}\,\forall_{x\in\Gamma}\,\forall_{r\in]0,1[}\,\exists_{y\in\mathbb{R}^n}\,B(y,\eta r)\subset B(x,r)\,\wedge\,B(x,\eta r)\cap\overline{\Gamma}=\emptyset,$$

where the notation $B(z,s)$ *means the closed ball centred at* z *with radius* s*.*

Lemma 2.4 *Given* $L \in \mathbb{N}_0$*, there are, for each* $\gamma \in \mathbb{N}_0^n$ *such that* $|\gamma| \leq L$*,* C^∞*-functions* ψ_γ *with support in the open ball* $\overset{\circ}{B}(0,1)$ *and satisfying the following property:*

$$\forall \beta,\gamma \in \mathbb{N}_0^n \text{ with } |\beta|,|\gamma| \leq L,\ \int_{\mathbb{R}^n} x^\beta \psi_\gamma(x)dx = \delta_{\beta\gamma},$$

where $\delta_{\beta\gamma}$ *stands for the Kronecker symbol.*

For a proof, see [13, p. 665].

Proposition 2.5 *Let* Ω *be a bounded domain such that* $\partial\Omega$ *has u.M.d.* D *and satisfies the ball condition. Then*

$$s < \frac{n-D}{p} \Rightarrow A^s_{pq}(\Omega) = \overset{\circ}{A}{}^s_{pq}(\Omega) \text{ for any } A \in \{B,F\}.$$

Proof. *Step 1.* We deal first with the case $A = B$. Moreover, in view of Proposition 2.2, we can now assume that $s \leq \sigma_p \equiv n(1/p-1)_+$.

Of course, Step 1 of the proof of Proposition 2.2 also applies here.

Step 2. We take a family

$$\{\varphi_{jm} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$$

of smooth functions in $\mathbb{R}^n$ as in (4) and prove the following:

"Given any $f \in \mathcal{S}$, there exists $(g_j)_j \in \mathcal{S}_\Omega$ such that

$$\left\| f - g_j | B^s_{pq}(\mathbb{R}^n)\right\| \leq c\, 2^{j(s-(n-d)/p)},\ \text{large } j\text{ ”}, \tag{10}$$

where $c > 0$ is a constant and d can be arbitrarily fixed such that $d > D$.

We start as in the corresponding part of Step 2 of the proof of Proposition 2.2, arriving at formula (7), where

$$a_{jm} \equiv c(K,f)^{-1} 2^{-j(s-n/p)} \varphi_{jm} f$$

are $(s,p)_{K,-1}$-atoms. However, as now we are assuming $s \leq \sigma_p$, in order to have atomic representations one must be sure to dispose of $(s,p)_{K,L}$-atoms for a fixed $L \geq [\sigma_p - s]$ (at least for $j > 0$ — and we can always assume that j is positive) — cf. [10, Th. 13.8 on p. 75]. The construction that follows is similar to what was done in [13, pp. 665-666] (for notations, please check with Step 2 of the proof of Proposition 2.2).

For each B_{jm} intersecting $(\overset{\circ}{\partial\Omega})_{2^{-j}}$, fix an element of $\partial\Omega$, x_{jm} say, at a distance less than 2^{-j} of B_{jm}. Clearly, there is a constant $c_1 > 0$ such that $B_{jm} \subset B(x_{jm}, c_1 2^{-j})$. Since we are concerned with large j only, we can assume that $0 < c_1 2^{-j} < 1$, so that, from the fact $\partial\Omega$ satisfies the ball condition, there exists $y_{jm} \in \mathbb{R}^n$ such that $B(y_{jm}, \eta c_1 2^{-j}) \subset B(x_{jm}, c_1 2^{-j})$ and $B(y_{jm}, \eta c_1 2^{-j}) \cap \partial\Omega = \emptyset$, where $0 < \eta < 1$ is as in Definition 2.3. Obviously, we can also say that $\mathrm{dist}(B(y_{jm}, \eta c_1 2^{-j-1}), \partial\Omega) \geq \eta c_1 2^{-j-1}$.

Fix $L \geq [\sigma_p - s]$ (with $L \in \mathbb{N}_0$) and let ψ_γ, with $\gamma \in \mathbb{N}_0^n$ and $|\gamma| \leq L$, be the functions whose existence was guaranteed in Lemma 2.4. Define, for each j, m as above,

$$d_\gamma^{jm} \equiv \int_{\mathbb{R}^n} x^\gamma a_{jm}(\eta c_1 2^{-j-1} x + y_{jm}) dx, \ \gamma \in \mathbb{N}_0^n \text{ with } |\gamma| \leq L,$$

and

$$\tilde{a}_{jm}(z) = a_{jm}(z) - \sum_{|\gamma| \leq L} d_\gamma^{jm} \psi_\gamma((\eta c_1)^{-1} 2^{j+1}(z - y_{jm})), \ z \in \mathbb{R}^n.$$

It is easy to see that

$$\int_{\mathbb{R}^n} ((\eta c_1)^{-1} 2^{j+1}(z - y_{jm}))^\beta \tilde{a}_{jm}(z) dz = 0, \ \beta \in \mathbb{N}_0^n \text{ with } |\beta| \leq L,$$

and, consequently (by Newton's binomial formula),

$$\int_{\mathbb{R}^n} z^\beta \tilde{a}_{jm}(z) dz = 0, \ \beta \in \mathbb{N}_0^n \text{ with } |\beta| \leq L.$$

This means that each $\tilde{a}_{jm}$ has the required moment conditions for the atoms in the atomic representations of functions of $B^s_{pq}(\mathbb{R}^n)$. Actually, it is not difficult to see that there exists a positive constant c_2 such that $c_2 \tilde{a}_{jm}$ is a $(s,p)_{K,L}$-atom.

Define, for each j as above,

$$h_j \equiv \sum_m (c(K,f)2^{j(s-n/p)}c_2^{-1})(c_2\tilde{a}_{jm}), \tag{11}$$

where, as in the proof of Proposition 2.2, the sum $\sum_m$ is taken over all $m \in \mathbb{Z}^n$ such that B_{jm} intersects $(\overset{\circ}{\partial\Omega})_{2^{-j}}$. From the hypotheses and choices that have been made, each such h_j belongs to $\mathcal{S}$ and equals $\sum_m \varphi_{jm} f$ on $(\partial\Omega)_{\eta c_1 2^{-j-1}}$. Since the latter sum equals f on $(\overset{\circ}{\partial\Omega})_{2^{-j}}$, then there is a positive constant c_3 such that

$$h_j = f \text{ on } (\overset{\circ}{\partial\Omega})_{c_3 2^{-j}}.$$

Define now $g_j \equiv f - h_j$. We have $g_j \in \mathcal{S}_\Omega$ and, since the right-hand side of (11) is an atomic representation for h_j in $B^s_{pq}(\mathbb{R}^n)$, we can write, on the basis of Theorem 13.8 of [10, p.75], that

$$\begin{aligned} \|f - g_j | B^s_{pq}(\mathbb{R}^n)\| &\leq c_4(\sum_m |c(K,f)2^{j(s-n/p)}c_2^{-1}|^p)^{1/p} \\ &\leq c_4 c_2^{-1} c(K,f) 2^{j(s-n/p)} (\sum_m 1)^{1/p} \\ &\leq c\, 2^{j(s-(n-d)/p)}, \text{ large } j, \end{aligned}$$

arguing as in the last part of Step 2 in the proof of Proposition 2.2.

Step 3. Note now that, if $s < (n-D)/p$, it is possible to choose $d > D$ such that $s < (n-d)/p$. Using this d in (10) we see we have proved that any $f \in \mathcal{S}$ is the limit, in $B^s_{pq}(\mathbb{R}^n)$, of the sequence $(g_j)_j \subset \mathcal{S}_\Omega$. By Step 1 we can thus conclude that

$$B^s_{pq}(\Omega) = \overset{\circ}{B}{}^s_{pq}(\Omega) \tag{12}$$

for $s < (n-D)/p$.

Step 4. It remains to deal with the case $A = F$. Since $B^s_{p,\min\{p,q\}}(\Omega) \hookrightarrow F^s_{pq}(\Omega)$ (see [9, p. 47] for the case $\Omega = \mathbb{R}^n$ and take advantage of the definition by restriction for the case of general bounded Ω) and (12) holds with $\min\{p,q\}$ instead of q, the conclusion follows by applying Lemma 2.1. $\square$

There is an interesting consequence of Proposition 2.5 expressed in terms of the fashionable notion of d-set. We recall the definition (taken from [6, pp. 28-33]) of the latter first.

Definition 2.6 *Let* Γ *be a non-empty closed subset of* $\mathbb{R}^n$ *and* $d \in]0,n]$. Γ *is said to be a* d*-set if*

$$\exists c_1, c_2 > 0 : \forall \gamma \in \Gamma, \forall r \in]0,1],\ c_1 r^d \leq \mathcal{H}^d(B(\gamma, r) \cap \Gamma) \leq c_2 r^d,$$

where $\mathcal{H}^d$ *denotes the* d*-dimensional Hausdorff measure on* $\mathbb{R}^n$ *and* $B(\gamma, r)$ *stands for the closed ball centred at* γ *and with radius* r.

For the definition and properties of Hausdorff measures and dimensions, see, for example, [3] or [7].

Corollary 2.7 *Let* Ω *be a bounded domain such that* $\partial\Omega$ *is a* d*-set, for some* $d < n$. *Then*

$$s < \frac{n-d}{p} \Rightarrow A^s_{pq}(\Omega) = \overset{\circ}{A}{}^s_{pq}(\Omega) \text{ for any } A \in \{B, F\}.$$

Proof. As remarked in [10, p. 6], $\partial\Omega$ being a d-set implies its Minkowski dimension (in particular, its upper Minkowski dimension) coincides with its Hausdorff dimension, that is, is d. Therefore, in order to finish the proof by applying Proposition 2.5 it is enough to guarantee that any d-set, $d < n$, satisfies the ball condition. This is indeed the case: a direct proof can be seen in [2], though, as we noticed *a posteriori*, this can also be realized in an indirect way using, for example, Proposition 2 in [5, p. 288]. □

3 Converse results

We would like to have a converse result to Corollary 2.7, and this goal is partially attained at the end of this section. The technique used depends on trace results, not always available in the literature, so we start with some considerations and results about traces.

We take the notion of trace $\mathrm{tr}_\Gamma f$ for functions f of $A^s_{pq}(\mathbb{R}^n)$ ($A \in \{B, F\}$) on d-sets Γ as in [10, pp. 138-139]: $\mathrm{tr}_\Gamma \varphi = \varphi|_\Gamma$, that is, is defined pointwise, when $\varphi \in \mathcal{S}$; tr_Γ is defined by completion for all remaining functions of $A^s_{pq}(\mathbb{R}^n)$ whenever it is possible to find $c > 0$ such that

$$\|\mathrm{tr}_\Gamma \varphi | L_p(\Gamma)\| \leq c \, \|\varphi | A^s_{pq}(\mathbb{R}^n)\|, \ \varphi \in \mathcal{S}$$

($L_p(\Gamma)$ considered with respect to the measure $\mathcal{H}^d|_\Gamma$). In particular, we are not claiming that the trace always exists! When it does — that is, when the above approach succeeds — we shall say that "$\mathrm{tr}_\Gamma A^s_{pq}(\mathbb{R}^n)$ exists". Note that, by definition, with this phrase we are also saying that the trace operator

$$\mathrm{tr}_\Gamma : A^s_{pq}(\mathbb{R}^n) \longrightarrow L_p(\Gamma)$$

is linear and bounded.

Lemma 3.1 *If Γ is a d-set, $d < n$, then*

$$s > \frac{n-d}{p} \Rightarrow \mathrm{tr}_\Gamma A^s_{pq}(\mathbb{R}^n) \text{ exists, } A \in \{B, F\}.$$

Proof. From Step 1 of the proof of Theorem 18.6 in [10, p. 139], $\mathrm{tr}_\Gamma B^{(n-d)/p}_{pq}(\mathbb{R}^n)$ exists if $q \leq \min\{1, p\}$. Since we are assuming that $s > (n-d)/p$, the result of the lemma follows then by using elementary embeddings between function spaces (cf., for example, [9, p. 47]). $\square$

For density results of $C^\infty_0(\mathbb{R}^n \setminus \Gamma)$ in the kernel of the trace operator, cf. [1, p. 281] and [12, Th. 1]. However, this will not be of use to us. Here, we need to deal with the question of existence of a trace $\mathrm{Tr}_{\partial\Omega} f$ on $\partial\Omega$ for functions f of $A^s_{pq}(\Omega)$, at least for some spaces $A^s_{pq}(\Omega)$.

Recall that $H^s_p(\mathbb{R}^n) \equiv F^s_{p2}(\mathbb{R}^n)$, $s > 0$, $p > 1$, are spaces of Bessel potentials and define $H^s_p(\Omega)$, $s > 0$, $p > 1$, as being equal to $F^s_{p2}(\Omega)$ — so, it is defined by restriction.

The following notion is taken from [13], with modifications:

Definition 3.2 *A domain Ω is said to be interior regular if*

$$\exists_{c>0} \, \forall_{x \in \partial\Omega} \, \forall_{\text{cube } Q \text{ centred at } x \text{ with side length } \leq 1} \, |\Omega \cap Q| \geq c \, |Q|.$$

It's easy to see that this is equivalent to saying that the domain Ω is a n-set, in the sense of [6, p. 205] (which is the same as saying that Ω is a n-set in the sense of Definition 2.6 above applied to $\Gamma = \Omega$, except that here we don't require Γ to be closed)

Proposition 3.3 *Let Ω be an interior regular domain such that $\partial\Omega$ is a d-set. If $p > 1$ and $s > (n-d)/p$ then*

$f \in H_p^s(\Omega) \Rightarrow$ *all* $u \in H_p^s(\mathbb{R}^n)$ *such that* $u|_\Omega = f$ *have the same trace on* $\partial\Omega$.

Proof. *Step 1.* We start by observing that, although not explicitly mentioned in the assertion of the proposition, it must be $d < n$: this follows from the fact $|\partial\Omega| = 0$, which in turn is implied by the interior regularity of Ω and Proposition 1 in [6, p. 205].

Step 2. In this and the following steps of the present proof, when considering $H_p^s(\mathbb{R}^n)$ we are always assuming that $p > 1$ and $s > (n-d)/p$.

Here we remark that, for any $u \in H_p^s(\mathbb{R}^n)$, we can equivalently define $\mathrm{tr}_{\partial\Omega} u$ by $\bar{u}$ given by

$$\bar{u}(x) \equiv \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)dy \tag{13}$$

at the points $x \in \partial\Omega$ where the limit exists. In order to see this, first we note that, from Theorem 1 in [6, p. 182] and the definition of Besov spaces on d-sets used there (cf. [6, p. 123]), the operator $u \mapsto \bar{u}$ maps $H_p^s(\mathbb{R}^n)$ linearly and boundedly into $L_p(\partial\Omega)$ (in particular, $\bar{u}$ makes sense $\mathcal{H}^d$-a.e. in $\partial\Omega$). Then the conclusion easily follows by using Lemma 3.1 and the facts that $\bar{u} = u|_{\partial\Omega} = \mathrm{tr}_{\partial\Omega} u$ when $u \in \mathcal{S}$ and that $\mathcal{S}$ is dense in $H_p^s(\mathbb{R}^n)$ (this type of argument is borrowed from [6, VIII.1.3, p. 211]).

Step 3. As the previous one, this is also a preliminary step to the proof of the proposition, though a longer one.

We take advantage of the fact $H_p^s(\mathbb{R}^n)$ is a space of Bessel potentials, so that any $u \in H_p^s(\mathbb{R}^n)$ can be written as the convolution $u = G_s * g$, for some $g \in L_p(\mathbb{R}^n)$, where G_s stands for the Bessel kernel of order s (see, for example, [1, pp. 9-11]). We can even say, due to Proposition 1 in [6, p. 151] (and the preceding step), that, for the range of parameters we are considering,

$$\mathrm{tr}_{\partial\Omega} u = \bar{u} = (G_s * g)|_{\partial\Omega} \quad \mathcal{H}^d\text{-a.e. in } \partial\Omega, \tag{14}$$

where the integral defining the convolution is absolutely convergent $\mathcal{H}^d$-a.e..

On the other hand, Theorem 3.2 of [8] assures us that if $v \in L_p(\mathbb{R}^n)$, $p > 1$, and $v \geq 0$, then, for $\mathcal{H}^d$-almost all x_0 in $\partial\Omega$, there exists $E \subset \mathbb{R}^n$ such that

$$\lim_{\substack{x \to x_0 \\ x \notin E}} (G_s * v)(x) = (G_s * v)(x_0) \tag{15}$$

and

$$C_{s,p}(E \cap \overset{\circ}{B}(x_0, r)) = o(r^d) \text{ as } r \to 0, \tag{16}$$

where $C_{s,p}$ stands for capacity and is defined in the following way:

$$\forall A \subset \mathbb{R}^n, C_{s,p}(A) \equiv \inf_h \int_{\mathbb{R}^n} h^p(x)dx,$$

where the infimum is taken over all $h \in L_p(\mathbb{R}^n)$ such that $h \geq 0$ and $(G_s * h)(x) \geq 1$ for all $x \in A$. It can, moreover, be seen that it is possible to choose E above as a (Lebesgue) measurable subset of $\mathbb{R}^n$. We shall consider such a choice in the sequel.

From (16) it follows that

$$|E \cap B(x_0, r)| = o(r^n) \text{ as } \lambda \to 0. \tag{17}$$

To prove this, we separate in two cases:

Case $s < n/p$:

By Sobolev embedding theorem (cf., e.g., [1, p. 14] and the references given there), we have $H_p^s(\mathbb{R}^n) \hookrightarrow L_q(\mathbb{R}^n)$ for $q = (np)/(n - sp)$. Hence, for all $h \in L_p(\mathbb{R}^n)$ with $h \geq 0$ and $(G_s * h)(x) \geq 1$ for all $x \in E \cap \overset{\circ}{B}(x_0, r)$,

$$\begin{aligned} |E \cap B(x_0, r)| &\leq \int_{E \cap B(x_0,r)} (G_s * h)(x)dx \\ &\leq \|G_s * h|L_q(\mathbb{R}^n)\| |E \cap B(x_0, r)|^{1/q'} \\ &\leq c_1 \|G_s * h|H_p^s(\mathbb{R}^n)\| |E \cap B(x_0, r)|^{1/q'} \\ &= c_1 \|h|L_p(\mathbb{R}^n)\| |E \cap B(x_0, r)|^{1/q'}, \end{aligned}$$

for some positive constant c_1, so that

$$|E \cap B(x_0, r)|^{p/q} \leq c_1^p C_{s,p}(E \cap \overset{\circ}{B}(x_0, r)).$$

Formula (17) then follows from this and (16), due to the particular value considered for q and the hypothesis $s > (n-d)/p$.

Case $s \geq n/p$:

If $s = n/p$, Sobolev embedding theorem gives now $H_p^s(\mathbb{R}^n) \hookrightarrow L_q(\mathbb{R}^n)$ for all $q \geq p$ ($q < \infty$, of course — recall we never consider infinite values for such parameters). However, the same result holds also for $s > n/p$, due to elementary embeddings, so that we can deal simultaneously with all possibilities of $s \geq n/p$. Formula (17) now follows as before, in case $s < n/p$: in the last part it is even easier, because now any large q is at our disposal and it is enough to choose $q \geq (np)/d$.

With the help of (17) we can, in a straightforward way, obtain conclusions (15) and (16) (and again (17)) for any complex $g \in L_p(\mathbb{R}^n)$ (instead of $v \in L_p(\mathbb{R}^n)$, $v \geq 0$) by decomposing into the real and imaginary parts and the positive and negative parts of these. However, as we shall see, we really need the original formulation of [8] for non negative functions in $L_p(\mathbb{R}^n)$.

Step 4. We come now to the proof proper of the proposition being considered.

Given $f \in H_p^s(\Omega)$ and any $u \in H_p^s(\mathbb{R}^n)$ such that $u|_\Omega = f$, by Step 3 (see, in particular, the conclusion (14)) we can find $g \in L_p(\mathbb{R}^n)$ such that $u = G_s * g$ and

$$\mathrm{tr}|_{\partial\Omega} u(x_0) = (G_s * g)(x_0) \tag{18}$$

for $\mathcal{H}^d$-almost all x_0 in $\partial\Omega$.

In what follows, v will denote either one of $(\Re g)_+$, $(\Re g)_-$, $(\Im g)_+$, $(\Im g)_-$. Of course, also $G_s * v \in H_p^s(\mathbb{R}^n)$, so that, by (14) and Step 2, we can write

$$(G_s * v)(x_0) = \lim_{r\to 0} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} (G_s * v)(x)dx \tag{19}$$

for $\mathcal{H}^d$-almost all x_0 in $\partial\Omega$.

We note that it is clearly possible to find a set $S \subset \partial\Omega$ of $\mathcal{H}^d$-measure 0 such that, for any $x_0 \in \partial\Omega \setminus S$, (18) and (19) hold and there is a set E, common to all four possibilities for our v, such that (15), (16) and (17) hold, as well as the limit result

$$\lim_{r\to 0} \frac{|E^c \cap B(x_0,r)|}{|B(x_0,r)|} = 1,$$

easily implied by (17). For such a x_0 it is then not difficult to see that, indeed,

$$(G_s * v)(x_0) = \lim_{r\to 0} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)\cap E^c} (G_s * v)(x)dx. \tag{20}$$

This and (19), in turn, imply that

$$\lim_{r\to 0} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)\cap E} (G_s * v)(x)dx = 0,$$

which, being valid for all four possibilities considered for our v, allows, together with the hypothesis of interior regularity for $\partial\Omega$, to conclude that

$$\lim_{r\to 0} \frac{1}{|B(x_0,r)\cap\Omega|} \int_{B(x_0,r)\cap\Omega\cap E} (G_s * g)(x)dx = 0. \tag{21}$$

The interior regularity of Ω also allows, together with the ingredients that led to (20), to obtain

$$\lim_{r\to 0} \frac{1}{|B(x_0,r)\cap\Omega|} \int_{B(x_0,r)\cap\Omega\cap E^c} (G_s * g)(x)dx = (G_s * g)(x_0).$$

The latter and (21) show that

$$(G_s * g)(x_0) = \lim_{r\to 0} \frac{1}{|B(x_0,r)\cap\Omega|} \int_{B(x_0,r)\cap\Omega} (G_s * g)(x)dx.$$

Combining with (18) and the hypotheses made on u at the beginning of the present step, we get

$$\mathrm{tr}_{\partial\Omega} u(x_0) = \lim_{r\to 0} \frac{1}{|B(x_0,r)\cap\Omega|} \int_{B(x_0,r)\cap\Omega} f(x)dx \tag{22}$$

for $\mathcal{H}^d$-almost all x_0 in $\partial\Omega$. Since the right-hand side of (22) is independent of the particular u considered, the proof is finished. □

Remark 3.4 **(i)** *The main ideas for the preceding proof are taken from the proofs of Proposition 2 in [6, pp. 206-208] or Theorem 1 in [14, pp. 121-122].*

(ii) *As a bonus we got, in the course of the preceding proof, the representation (13) for* $tr_{\partial\Omega}u$, $u \in H^s_p(\mathbb{R}^n)$, *and the representation (22) for the common trace on* $\partial\Omega$ *of all* $u \in H^s_p(\mathbb{R}^n)$ *such that* $u|_\Omega = f$, *for some given* $f \in H^s_p(\Omega)$, *in both cases under the conditions considered. After the next definition, (22) will also be a representation for the trace* $Tr_{\partial\Omega}f$ *of* f *on* $\partial\Omega$. *We want nevertheless to point out that, though they might ease some considerations below, those representations are really not needed in what follows, and so they are not used hereafter.*

Definition 3.5 *Let* Ω *be an interior regular domain such that* $\partial\Omega$ *is a d-set. If* $p > 1$ *and* $s > (n-d)/p$, *we define the trace of* $f \in H^s_p(\Omega)$ *on* $\partial\Omega$ *as the element* $Tr_{\partial\Omega}f$ *of* $L_p(\partial\Omega)$ *given by*

$$Tr_{\partial\Omega}f \equiv tr_{\partial\Omega}u,$$

where u *is any element of* $H^s_p(\mathbb{R}^n)$ *such that* $u|_\Omega = f$ *(we recall that* $L_p(\partial\Omega)$ *is considered with respect to the measure* $\mathcal{H}^d|_{\partial\Omega}$*).*

By Proposition 3.3, this definition makes sense.

Proposition 3.6 *Let* Ω *be an interior regular domain such that* $\partial\Omega$ *is a d-set. If* $p > 1$ *and* $s > (n-d)/p$, *the trace operator*

$$Tr_{\partial\Omega} : H^s_p(\Omega) \to L_p(\partial\Omega)$$

is linear and continuous.

Proof. Both linearity and boundedness are obvious from the corresponding properties of $tr_{\partial\Omega}$ and the definition of $H^s_p(\Omega)$. $\square$

Proposition 3.7 *Let* Ω *be an interior regular domain such that* $\partial\Omega$ *is a d-set. Then, for any* $A \in \{B, F\}$,

$$A^s_{pq}(\Omega) \neq \mathring{A}^s_{pq}(\Omega) \tag{23}$$

if either

(i) $p > 1$ *and* $s > \frac{n-d}{p}$ *or*

(ii) $p \leq 1$ *and* $s > \frac{n}{p} - d$.

Proof. *Step 1.* We deal first with (i) and $A^s_{pq}(\Omega) = H^s_p(\Omega)$.

On one hand, there are elements of $H^s_p(\Omega)$ with a non-zero trace on $\partial\Omega$: for example, any $\varphi|_\Omega$ such that $\varphi \in C^\infty_0(\mathbb{R}^n)$ and $\varphi \equiv 1$ on $B(\gamma, 1)$, for some $\gamma \in \partial\Omega$.

On the other hand, if $H^s_p(\Omega) = \overset{\circ}{H}{}^s_p(\Omega)$ were true, any $f \in H^s_p(\Omega)$ could be arbitrarily approximated in $H^s_p(\Omega)$ by functions in $C^\infty_0(\Omega)$, the trace of which is zero. Hence, by the continuity asserted in Proposition 3.6, $\mathrm{Tr}_{\partial\Omega} f = 0$ too.

We got a contradiction and therefore $H^s_p(\Omega) \neq \overset{\circ}{H}{}^s_p(\Omega)$ in case (i).

Step 2. Next we deal with both (i) and (ii) for $A = F$.

In case (i) one simply has to note that we can choose $\varepsilon > 0$ such that $s - \varepsilon > (n-d)/p$ and $F^s_{pq}(\Omega) \hookrightarrow F^{s-\varepsilon}_{p2}(\Omega) = H^{s-\varepsilon}_p(\Omega)$ (see [9, p.47] for the case of $\Omega = \mathbb{R}^n$ and take advantage of the definition by restriction for the case of general Ω). Lemma 2.1 and Step 1 do the rest.

In case (ii) it is possible to choose $p_1 > 1 \geq p$ and $s_1 > (n-d)/p_1$ such that $s - n/p = s_1 - n/p_1$. If $F^s_{pq}(\Omega) = \overset{\circ}{F}{}^s_{pq}(\Omega)$ were true, then, since $F^s_{pq}(\Omega) \hookrightarrow F^{s_1}_{p_1 q}(\Omega)$ (see [9, p. 129] for the case $\Omega = \mathbb{R}^n$), Lemma 2.1 would imply that $F^{s_1}_{p_1 q}(\Omega) = \overset{\circ}{F}{}^{s_1}_{p_1 q}(\Omega)$, contradicting what we have already obtained in case (i).

Step 3. Finally, we get (i) and (ii) for $A = B$ by choosing $\varepsilon > 0$ such that $s - \varepsilon > (n-d)/p$ (in case (i)) or $s - \varepsilon > n/p - d$ (in case (ii)) and $B^s_{pq}(\Omega) \hookrightarrow B^{s-\varepsilon}_{p,\min\{p,q\}}(\Omega) \hookrightarrow F^{s-\varepsilon}_{pq}(\Omega)$ (see again [9, p. 47] for the case $\Omega = \mathbb{R}^n$), and by applying the preceding step and Lemma 2.1 (once more). □

4 Examples

As we have just seen in Proposition 3.7, in the case $p \geq 1$ we have obtained a reasonable converse to Corollary 2.7. The case $p < 1$ is less satisfactory: combining the results just mentioned, there is a gap for s between $(n-d)/p$ and $n/p - d$ where neither (23) nor its opposite is asserted. The natural guess would, of course, be that (23) also holds in that gap. In the present section we want to describe classes of examples which point out in that direction.

Proposition 4.1 *Let ω be a bounded domain in $\mathbb{R}^n$ and Γ a d-set, $d < n$, such that $\Gamma \subset \omega$. Let $\Omega \equiv \omega \setminus \Gamma$. If $s > \sigma_p \equiv n(1/p-1)_+$, then*

$$s > \frac{n-d}{p} \Rightarrow A^s_{pq}(\Omega) \neq \overset{\circ}{A}{}^s_{pq}(\Omega) \text{ for any } A \in \{B, F\}.$$

Proof. *Step 1.* We first remark that, on the assumption $s > (n-d)/p$, $\mathrm{tr}_\Gamma A^s_{pq}(\mathbb{R}^n)$ cannot be zero (that it exists follows from Lemma 3.1). Indeed, any $\varphi \in C^\infty_0(\mathbb{R}^n)$ such that $\varphi \equiv 1$ on $B(\gamma, 1)$, for some $\gamma \in \Gamma$, is an element of $A^s_{pq}(\mathbb{R}^n)$ with a non-zero trace on Γ.

Step 2. Assume that the conclusion was false for such a Ω, that is,

$$A^s_{pq}(\Omega) = \overset{\circ}{A}{}^s_{pq}(\Omega) \text{ with } s > \sigma_p, \frac{n-d}{p}.$$

Let f be any function in $A^s_{pq}(\mathbb{R}^n)$. Then $f|_\Omega \in A^s_{pq}(\Omega)$ and therefore for each $\varepsilon > 0$ there would exist $\varphi \in C^\infty_0(\Omega)$ such that $\|f|_\Omega - \varphi \,|A^s_{pq}(\Omega)\| < \varepsilon$. By definition of this quasi-norm, it would also exist $g \in A^s_{pq}(\mathbb{R}^n)$ such that $g|_\Omega = f|_\Omega - \varphi$ and $\|g \,|A^s_{pq}(\mathbb{R}^n)\| < \varepsilon$. Consider $f^\Omega \equiv f - g$. We have

$$\|f - f^\Omega \,|A^s_{pq}(\mathbb{R}^n)\| < \varepsilon \tag{24}$$

and $f^\Omega|_\Omega = \varphi$. As a consequence, it would be possible to approach arbitrarily any given $f \in A^s_{pq}(\mathbb{R}^n)$ by functions (as f^Ω above) which vanish in a one-sided neighbourhood $(\overset{\circ}{\partial\Omega})_\delta$ of $\partial\Omega$, where $\delta > 0$ may change with the f^Ω, of course (for the definition of $(\overset{\circ}{\partial\Omega})_\delta$, see (6) and (1)).

Step 3. Note that from the hypotheses made about ω, Γ and Ω in the proposition, it follows that, for sufficiently small $\delta > 0$,

$$\Gamma_\delta \setminus \Gamma = \overset{\circ}{\Gamma}_\delta \equiv \Gamma_\delta \cap \Omega \subset (\overset{\circ}{\partial\Omega})_\delta,$$

so that f^Ω would also vanish in $\Gamma_\delta \setminus \Gamma$. Now, since $|\Gamma| = 0$ (because Γ is a d-set with $d < n$) and f^Ω is a regular distribution (because $f^\Omega \in A^s_{pq}(\mathbb{R}^n) \hookrightarrow L^{loc}_1(\mathbb{R}^n)$, as follows from the hypothesis $s > \sigma_p$ and the properties of the spaces in question — cf. [9]), f^Ω could be identified with a distribution in $A^s_{pq}(\mathbb{R}^n)$ vanishing in all Γ_δ.

Step 4. Consider $\varphi^\delta \in C_0^\infty(\mathbb{R}^n)$ equal to 1 in $\Gamma_{\delta/2}$ and with support in Γ_δ and define $\chi^\delta \equiv 1 - \varphi^\delta$. It is easily seen that χ^δ belongs to the Zygmund space $\mathcal{C}^\rho(\mathbb{R}^n)$ — cf. [9, p. 36] —, for any $\rho > 0$, and that $\chi^\delta f^\Omega = f^\Omega$. Hence χ^δ is a pointwise multiplier for $A^s_{pq}(\mathbb{R}^n)$ (see [9, pp. 140-141]) and, therefore, by multiplying it with any given sequence $(\psi_k)_{k\in\mathbf{N}} \in \mathcal{S}$ tending to f^Ω in $A^s_{pq}(\mathbb{R}^n)$ we see that f^Ω can be arbitrarily approximated in this space by $\mathcal{S}$-functions vanishing in Γ.

An application of Lemma 3.1 now shows that, on the assumption $s > (n-d)/p$, $\mathrm{tr}_\Gamma f^\Omega$ would be zero. Combining this with Step 2 we would get, under the same assumptions and again by Lemma 3.1, that any $f \in A^s_{pq}(\mathbb{R}^n)$ would have zero trace on Γ.

However, this contradicts Step 1 and therefore the conclusion of the proposition must be valid. □

Remark 4.2 **(i)** *In order to compare this result with Proposition 2.2 or Proposition 2.5, note that for smooth enough $\partial\omega$ ($\partial\omega$ with u.M.d. equal to $n-1$ is enough) and admissible $d \in [n-1, n[$ we get Ω in Proposition 4.1 such that the u.M.d. of $\partial\Omega$ is d.*

(ii) *If, instead of Lemma 3.1, we use Corollary 18.12 in [10, p. 142], we can even get, for the same class of sets Ω, $A^s_{pq}(\Omega) \neq \mathring{A}^s_{pq}(\Omega)$ for $s = (n-d)/p$, as long as we further assume that $q \le \min\{1, p\}$ (in the case $A = B$) or $p \le 1$ (in the case $A = F$) and (in both cases) $p > d/n$.*

Proposition 4.3 *For each $D \in [n-1, n[$ (with $n \ge 2$) there is a bounded simply connected domain Ω such that $\partial\Omega$ has u.M.d. equal to D and for which we have*

$$s > \frac{n-D}{p} \Rightarrow A^s_{pq}(\Omega) \neq \mathring{A}^s_{pq}(\Omega) \text{ for any } A \in \{B, F\}.$$

Proof. *Step 1.* Consider any D as above. According to Theorem 16.2, its proof and Remark 16.3 in [10, pp. 120-122], there is a continuous non-negative function h defined in the closed unit cube Q (centred at 0) in $\mathbb{R}^{n-1}$

such that $h = 0$ on $\partial\Omega$ and

$$\Gamma \equiv \{(x, h(x)) \in \mathbb{R}^n : x \in Q\}$$

is a D-set (in $\mathbb{R}^n$). Consider the bounded simply connected domain

$$\Omega \equiv \{(x, y) \in \mathbb{R}^n : x \in 2\,\overset{\circ}{Q} \wedge -1 < y < h(x)\},$$

where $2\,\overset{\circ}{Q}$ denotes a cube (in $\mathbb{R}^{n-1}$) with the same centre of $\overset{\circ}{Q}$ (the interior of Q) and twice its side length, and where h has been extended by 0 outside Q. Clearly, $\partial\Omega$ has u.M.d. equal to D (though it is not a D-set, unless $D = n - 1$).

Now assume that the conclusion of the proposition was false for such a Ω, that is,

$$A^s_{pq}(\Omega) = \overset{\circ}{A}{}^s_{pq}(\Omega) \text{ with } s > \frac{n - D}{p},$$

and, for any $f \in A^s_{pq}(\mathbb{R}^n)$, repeat the arguments given in Step 2 of the proof of Proposition 4.1 in order to show that it would be possible to approach f arbitrarily by functions $f^\Omega \in A^s_{pq}(\mathbb{R}^n)$ which vanish in a one-sided neighbourhood $(\overset{\circ}{\partial\Omega})_\delta$ of $\partial\Omega$ (with $\delta > 0$ depending on f^Ω).

Step 2. Consider any such f^Ω and assume $\delta < 1$ (which we can certainly do). It is easy to see that f^Ω also would vanish in the set

$$A_\delta \equiv \{(x, y) \in \mathbb{R}^n : x \in 2\,\overset{\circ}{Q} \wedge h(x) - \delta < y < h(x)\}$$

and, consequently, for each $0 < t < \delta$, the function $f^\Omega((x, y - t))$ would vanish in the set $(0, t) + A_\delta$, which contains $\{(x, h(x)) \in \mathbb{R}^n : x \in 2\,\overset{\circ}{Q}\}$.

We claim that, as $t \to 0$,

$$f^\Omega((x, y - t)) \to f^\Omega \text{ in } A^s_{pq}(\mathbb{R}^n). \tag{25}$$

In fact, given any $\varepsilon > 0$, there exists $\psi \in C_0^\infty(\Omega)$ such that

$$\|f^\Omega((x, y - t)) - \psi((x, y - t))|A^s_{pq}(\mathbb{R}^n)\| = \|f^\Omega - \psi|A^s_{pq}(\mathbb{R}^n)\| < \varepsilon,$$

so that we can write, with some positive constant c,

$$\|f - f^\Omega((x, y - t))|A^s_{pq}(\mathbb{R}^n)\| \le c\,(2\varepsilon + \|\psi - \psi((x, y - t))|A^s_{pq}(\mathbb{R}^n)\|).$$

Since, using the continuous embedding $\mathcal{S} \hookrightarrow A^s_{pq}(\mathbb{R}^n)$, we can easily estimate the latter quasi-norm by constant times ε, for small enough values of t, the claim follows.

Step 3. Since the compact set $\Gamma = \{(x, h(x)) \in \mathbb{R}^n : x \in Q\}$ and the closed set $((0,t) + A_\delta)^c$ are disjoint, there is a neighbourhood $\Gamma_{t'}$ of Γ all inside $(0,t) + A_\delta$, and therefore $f^\Omega((x, y-t))$ also would vanish in $\Gamma_{t'}$.

Now argue as in Step 4 of the proof of Proposition 4.1, with t' and $f^\Omega((x, y-t))$ in place of δ and f^Ω, respectively, and get $\mathrm{tr}_\Gamma f^\Omega((x, y-t)) = 0$ on the assumption $s > (n-D)/p$. Lemma 3.1 applied first to (25) and afterwards to the conclusion of Step 1 would then lead to $\mathrm{tr}_\Gamma f = 0$, again under the assumption $s > (n-D)/p$. However, since this would be true for any $f \in A^s_{pq}(\mathbb{R}^n)$, we would have got a contradiction with the fact that $\mathrm{tr}_\Gamma A^s_{pq}(\mathbb{R}^n)$ cannot be zero (cf. Step 1 of the proof of Proposition 4.1). □

Remark 4.4 *Remark 4.2(ii) also applies here, mutatis mutandis (but now there is no need for the assumption $p > D/n$).*

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Departamento de Matemática
Universidade de Aveiro
3810-193 AVEIRO
Portugal
E-mail: acaetano@mat.ua.pt