Sharp estimates of approximation numbers via growth envelopes

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Dedicated to Prof. Hans Triebel on the occasion of his 65th birthday

Abstract

We give sharp asymptotic estimates for the approximation numbers of the compact embedding $id : L_p(\log L)_a(\Omega) \longrightarrow B^{-1}_{\infty,\infty}(\Omega)$, $a > 0$, $n < p < \infty$, applying the newly developed tool of (growth) envelopes for function spaces.

Introduction

We present an application of our recently developed concept of envelopes in function spaces : this is a relatively simple tool for the study of rather complicated spaces, say, of Sobolev type H^s_p, or Besov type $B^s_{p,q}$, in '*limiting*' situations. Arising from the famous Sobolev embedding theorem it is, for instance, well-known that $B^{n/p}_{p,q} \hookrightarrow L_\infty$ if, and only if, $0 < p \le \infty$, $0 < q \le 1$; thus a natural question is in what sense the unboundedness of functions belonging to $B^{n/p}_{p,q}$ with $1 < q \le \infty$ (or $H^{n/p}_p$ with $1 < p < \infty$) can be qualified. Concentrating on this particular feature only, the concept of growth envelope functions $\mathcal{E}^X_{\mathsf{G}}$ 'measuring' the unboundedness of functions f belonging to some function space X of regular distributions is introduced by means of their non-increasing rearrangement $f^*(t)$,

$$\mathcal{E}^X_{\mathsf{G}}(t) = \sup_{\|f|X\|\le 1} f^*(t) , \quad t > 0.$$

We found rather simple and final answers characterising the unboundedness of functions in spaces like $B^s_{p,q}$ and H^s_p; in fact, the results contain an even finer description of this feature than measured by $\mathcal{E}^X_{\mathsf{G}}$ merely and cover far more settings than the above-described limiting ones; see [Har02], [Tri01], [CaM01]. There are some direct connections with classical results, for instance with the so-called fundamental function φ_X when X is rearrangement-invariant. More interesting, however, are the new and, in our opinion, elegant results for spaces like $B^s_{p,q}$ or H^s_p.

Analogously one can investigate limiting situations when questions of (un)boundedness of functions are replaced by inquiries about (almost) Lipschitz continuity; though there are immediate counterparts we do not pursue this point here further.

Our main goal in [Har02] (as well as in [Tri01], [CaM01]) was to obtain precise but simple characterisations of technically rather complicated (scales of) spaces in the sense described above. In the course of our studies it turned out that this also leads to a lot of interesting consequences. Naturally one arrives at Hardy-type

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inequalities, but there is also an interplay between envelope functions and lift operators in function spaces as well as applications to related questions of compactness; this latter subject will be explained in the present paper in a little detail. Our idea is twofold : on the one hand we give an immediate (though not obvious) application of our envelope results to estimates for approximation numbers which are surprisingly sharp. Secondly we close a gap concerning asymptotic estimates for approximation numbers in function spaces (characterising in that way the compactness of the corresponding natural embedding) which was not covered by earlier results so far. Roughly speaking, the upper estimate rests upon an envelope function argument whereas the corresponding lower one is obtained by a careful combination of earlier approximation number estimates and complex interpolation.

We collect the necessary background material first, present our main result in Section 2, whereas its proof is postponed to Section 3. Finally, we end this paper with a small collection of related results.

1 Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary $\partial\Omega$. Recall that for $1 < p < \infty$, $a \in \mathbb{R}$, the space $L_p(\log L)_a(\Omega)$ consists of all measurable functions $f : \Omega \longrightarrow \mathbb{C}$ such that

$$\left(\int_0^{|\Omega|} \left[(1 + |\log t|)^a \, f^*(t) \right]^p \, \mathrm{d}t \right)^{\frac{1}{p}} \tag{1}$$

is finite, where $f^*(t)$ denotes the non-increasing rearrangement of f, as usual. For our later purpose of estimating approximation numbers, however, we prefer a characterisation of $L_p(\log L)_a(\Omega)$ by extrapolation techniques; this was obtained by Edmunds and Triebel in [ET96, Thm. 2.6.2, p. 69]. For convenience we adopt the notation

$$\frac{1}{p^\sigma} \equiv \frac{1}{p} + \frac{\sigma}{n}, \tag{2}$$

where $0 < \sigma < \varepsilon$ and we shall always assume in the sequel $\varepsilon > 0$ to be sufficiently small such that $p^\sigma > 1$. Then

$$\left(\int_0^{\varepsilon} \left[\sigma^a \| f | L_{p^\sigma}(\Omega) \| \right]^p \frac{\mathrm{d}\sigma}{\sigma} \right)^{1/p} \tag{3}$$

defines an equivalent norm on $L_p(\log L)_{-a}(\Omega)$, $1 < p < \infty$, $a > 0$; cf. [ET96, Thm. 2.6.2, p. 69].

Recall that $B^s_{p,q}(\Omega)$ stand for the Besov spaces, where $s \in \mathbb{R}$ and $0 < p, q \leq \infty$; see, for example, Chapter 2 of [ET96] for definitions and properties, in particular for embedding relations between such spaces.

Recall also the notion of approximation numbers $a_k(T)$, $k \in \mathbb{N}$, of $T \in L(E, F)$, where $L(E, F)$ stands for the collection of all bounded linear operators acting from the quasi-Banach space E into the quasi-Banach space F:

$$a_k(T) \equiv \inf\{\|T - S\| : S \in L(E, F), \, \mathrm{rank}\, S < k\},$$

where $\mathrm{rank}\, S$ is the dimension of the range of S. There are two properties of these numbers which will be useful in the sequel:

(i) (monotonicity)

$$\|T\| = a_1(T) \geq a_2(T) \geq \ldots \geq 0;$$

(ii) (multiplicativity)

$$a_{k+l-1}(R \circ S) \leq a_k(R) a_l(S),$$

where $S \in L(E,F)$ and $R \in L(F,G)$, with G another quasi-Banach space.

We shall also need the notion of the (local) growth envelope function applied to the spaces $L_p(\log L)_a(\Omega)$, which, for simplicity, we define just as

$$\mathcal{E}_{\mathsf{G}}^{L_p(\log L)_a}(t) \equiv \sup\{f^*(t) : \|f|L_p(\log L)_a(\Omega)\| \leq 1\}$$

for all small positive numbers t; for general considerations concerning growth envelope functions, see [Har02], [Tri01].

The behaviour of such functions will be important to us only up to equivalence $\sim$, where $g(t) \sim h(t)$ means the existence of two positive constants c_1 and c_2 such that $c_1\ g(t)\ \leq h(t)\ \leq\ c_2\ g(t)$ for all t under consideration. By the way, the notion of equivalence $\sim$ will also be applied to sequences, that is, to functions of the natural variable k instead of the continuous variable t, and in whatever situation where the existence of constants as c_1 and c_2 independent of the variable in the formula is implied.
Finally, let us recall that p' stands for the conjugate of p (that is, satisfies $\frac{1}{p} + \frac{1}{p'} = 1$) and that, for convenience, log will be used instead of $\log_2$.

2 The main result

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary $\partial\Omega$.*

(i) *Assume $a > 0$, $n < p < \infty$ for $n \geq 2$, and $2 \leq p < \infty$ when $n = 1$. Then for all $k \in \mathbb{N}$,*

$$a_k\left(id : L_p(\log L)_a(\Omega) \longrightarrow B^{-1}_{\infty,\infty}(\Omega)\right)\ \sim\ k^{-\left(\frac{1}{n}-\frac{1}{p}\right)}\,(1+\log k)^{-a}\ . \tag{4}$$

(ii) *Assume $a > 0$, $1 < p < \frac{n}{n-1}$ for $n \geq 2$, and $1 < p \leq 2$ when $n = 1$. Then for all $k \in \mathbb{N}$,*

$$a_k\left(id : B^1_{1,1}(\Omega) \longrightarrow L_p(\log L)_{-a}(\Omega)\right)\ \sim\ k^{-\ \frac{1}{n}-\frac{1}{p'}}\ \ (1+\log k)^{-a}\ . \tag{5}$$

3 Proofs

We start with three auxiliary results which will be used in the proof of the theorem; we thus split the (otherwise rather long) argument for our main result in more handy pieces. Moreover, it becomes more obvious that way which different tools will be applied afterwards. Let all function spaces be defined on $\Omega \subset \mathbb{R}^n$ in the sequel unless otherwise stated; here $\Omega \subset \mathbb{R}^n$ stands for a bounded domain with C^∞ boundary $\partial\Omega$.

Proposition 2 [Har02, Prop. 5.1.2] *Let $1 < p < \infty$, and $a \in \mathbb{R}$. Then*

$$\mathcal{E}_{\mathsf{G}}^{L_p(\log L)_a}(t)\ \sim\ t^{-\frac{1}{p}}\,|\log t|^{-a}\,, \quad 0 < t < \frac{1}{2}\ . \tag{6}$$

Let $X(\Omega)$ be some function space of regular distributions with $X(\Omega) \not\hookrightarrow L_\infty(\Omega)$. Denote by $X^\nabla(\Omega) \subset X(\Omega)$ the subspace

$$X^\nabla(\Omega) = \{g \in D'(\Omega) \ : \ D^\alpha g \in X(\Omega),\ |\alpha| \le 1\} \tag{7}$$

normed by

$$\left\| g|X^\nabla(\Omega) \right\| \ = \ \sum_{|\alpha|\le 1} \| D^\alpha g|X(\Omega) \| \ . \tag{8}$$

Let C stand for the space of all complex-valued bounded uniformly continuous functions on $\overline{\Omega}$, equipped with the sup-norm as usual.

Proposition 3 *Let $X(\Omega)$, $X^\nabla(\Omega)$ be the spaces given by (7), (8). Assume $X(\Omega) \hookrightarrow B^{-1}_{\infty,\infty}(\Omega)$ and $X^\nabla(\Omega) \hookrightarrow C(\Omega)$. Let $\mathcal{E}^X_{\mathsf{G}}$ satisfy*

$$\sum_{k=0}^{\infty} 2^{-k} \ \frac{\mathcal{E}^X_{\mathsf{G}} \left(2^{-(k+J)n}\right)}{\mathcal{E}^X_{\mathsf{G}} \left(2^{-Jn}\right)} \ \le \ c \ , \tag{9}$$

for some number $C > 0$ and for all large $J \in \mathbb{N}$, and assume that there is a bounded (linear) lift operator L mapping $X(\Omega)$ into $X^\nabla(\Omega)$ such that its inverse L^{-1} exists and maps $C(\Omega)$ into $B^{-1}_{\infty,\infty}(\Omega)$. Then

$$a_k \left(id : X(\Omega) \longrightarrow B^{-1}_{\infty,\infty}(\Omega) \right) \ \le \ c\ k^{-\frac{1}{n}} \ \mathcal{E}^X_{\mathsf{G}} \left(k^{-1}\right) \ . \tag{10}$$

Proof (*Proposition 3*) : The proof essentially relies on a result of Carl and Stephani [CaS90, Thm. 5.6.1, p. 178] together with the definition of envelope functions. It can be obtained parallel to [Har02, Cor. 7.2.3] (dealing with entropy numbers), see also [Har01, 6.15]. One simply combines results on approximation numbers for compact embeddings in $C(\Omega)$ (as target space) with the properties of the operator L and its inverse L^{-1}, a lift argument for envelope functions (which requires (9)), and, finally, the multiplicativity of approximation numbers.

Proposition 4 *Let $1 < r_0 < r_1 \le 2$ be such that $n < r_1'$. Then*

$$a_k(B^1_{1,1}(\Omega) \hookrightarrow L_r(\Omega)) \ge c\, k^{-(\frac{1}{n} - \frac{1}{r'})} \tag{11}$$

for $r \in [r_0, r_1]$, with $c > 0$ independent of r and k.

Proof (*Proposition 4*) : It follows by inspection of Step 3 of the proof of Theorem 3.3.4 in [ET96, pp. 123-125], and taking care of $L_r(\Omega) \hookrightarrow B^0_{r,2}(\Omega)$, that $id^l : l_1^{N_j} \to l_r^{N_j}$ can be decomposed (maybe after some translation and rescaling arguments) as

$$l_1^{N_j} \ \xrightarrow{A} \ B^1_{1,1}(\Omega) \ \xrightarrow{id^L} \ L_r(\Omega) \ \xrightarrow{B} \ l_r^{N_j},$$

i.e.

$$id^l = B \circ id^L \circ A, \tag{12}$$

where $N_j = 2^{jn}$, $j \in \mathbb{N}$, and where the bounded linear operators A and B can be chosen independently of $r \in [r_0, r_1]$. Furthermore, [ET96, pp. 123-125] gives us the estimates

$$\|A\| \le c_1 2^{j(1-n)} \quad \text{and} \quad \|B\| \le c_2\ 2^{j\frac{n}{r}},$$

where the positive constants c_1 and c_2 are independent of j, though c_2 can depend on r. However, writing $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$, for some $\theta \in]0,1[$, one has the complex interpolation formulae

$$[L_{r_0}(\Omega), L_{r_1}(\Omega)]_\theta = L_r(\Omega) \quad \text{and} \quad [l_{r_0}^{N_j}, l_{r_1}^{N_j}]_\theta = l_r^{N_j},$$

with equality of norms too (cf. [Tri78, 1.18.1,1.18.4]), so that

$$\|B : L_r(\Omega) \to l_r^{N_j}\| \le c_2^{1-\theta}(r_0)\, 2^{(1-\theta)j\frac{n}{r_0}}\, c_2^\theta(r_1)\, 2^{\theta j \frac{n}{r_1}} = c_3 2^{j\frac{n}{r}},$$

where now $c_3 > 0$ is independent of j and r.
The multiplicativity of the approximation numbers applied to (12) then gives

$$a_k(id^L) \ge \|A\|^{-1}\|B\|^{-1} a_k(id^l) \ge c_1^{-1} c_3^{-1} 2^{-j(1-\frac{n}{r'})} a_k(id^l)$$

for all $k \in \mathbb{N}$. Choosing $k \le \frac{N_j}{4} = 2^{jn-2}$, one can write

$$a_k(id^L) \ge c_4 2^{-j(1-\frac{n}{r'})}$$

where c_4 is independent of k, j and r (cf. [ET96, Corol. 3.2.3]).
Since for each $k \in \mathbb{N}$ there is (exactly) one $j \in \mathbb{N}$ such that $2^{(j-1)n-2} < k \le 2^{jn-2}$, (11) now follows easily by standard arguments.

Proof (*Theorem 1*) : Step 1. We first verify that (5) is an immediate consequence of (4) and duality arguments; note that $[L_p(\log L)_a(\Omega)]' = L_{p'}(\log L)_{-a}(\Omega)$, $1 < p < \infty$, $a \in \mathbb{R}$, see [BR80, Thm. 8.4]. On the other hand, $\left[B^1_{1,1}(\mathbb{R}^n)\right]' = B^{-1}_{\infty,\infty}(\mathbb{R}^n)$, cf. [Tri83, Thm. 2.11.2 (i), p. 178]; for the counterpart on Ω, one could start with spaces $\widetilde{B^s_{p,q}}$ – containing functions $f \in B^s_{p,q}$ with supp $f \subset \overline{\Omega}$ – first and extend afterwards. Thus taking (i) for granted at the moment, the duality result for approximation numbers, see [CaS90, Prop. 2.5.4, p. 80] and [ETy86], yields (ii),

$$a_k\left(L_p(\log L)_a(\Omega) \hookrightarrow B^{-1}_{\infty,\infty}(\Omega)\right) \sim a_k\left(B^1_{1,1}(\Omega) \hookrightarrow L_{p'}(\log L)_{-a}(\Omega)\right). \tag{13}$$

Step 2. We show the upper estimate in (4) and benefit from the preceding Propositions 2, 3. As $\mathcal{E}_{\mathsf{G}}^{L_p(\log L)_a}(t)$ given by (6) satisfies (9),

$$\sum_{k=0}^{\infty} 2^{-k}\, \frac{\mathcal{E}_{\mathsf{G}}^X\left(2^{-(k+J)n}\right)}{\mathcal{E}_{\mathsf{G}}^X\left(2^{-Jn}\right)} \sim \sum_{k=0}^{\infty} 2^{-k(1-\frac{n}{p})} \left(\frac{k+J}{J}\right)^{-a} \le c$$

for $p > n$, $a \in \mathbb{R}$, we can apply Proposition 3 with $X = L_p(\log L)_a$, $X^\nabla = H^1_p(\log L)_a$. The existence of the bounded linear lift operator $L : L_p(\log L)_a(\Omega) \longrightarrow H^1_p(\log L)_a(\Omega)$ is covered by [ET96, Thm. 2.6.3, p. 79], whereas the additional assumption on L^{-1} to map $C(\Omega)$ into $B^{-1}_{\infty,\infty}(\Omega)$ is a consequence of restriction-extension procedures and the usual lift operator $I_\sigma : B^s_{p,q} \longrightarrow B^{s-\sigma}_{p,q}$ in $\mathbb{R}^n$. Alternatively one can use regular elliptic differential operators adapted to Ω; see [Tri78, Thm. 4.9.2, p. 335] for the case $1 < p < \infty$, $1 \le q \le \infty$, and [Tri83, Thm. 4.3.4, p. 235] for the extensions to $0 < p, q \le \infty$. Hence (6) and (10) imply

$$a_k\left(id : L_p(\log L)_a(\Omega) \longrightarrow B^{-1}_{\infty,\infty}(\Omega)\right) \le c\, k^{-\left(\frac{1}{n}-\frac{1}{p}\right)} (1+\log k)^{-a}. \tag{14}$$

Step 3. We prove the inequality converse to (14), i.e. the existence of some $c > 0$ such that for all $k \in \mathbb{N}$,

$$a_k\left(id : L_p(\log L)_a(\Omega) \longrightarrow B^{-1}_{\infty,\infty}(\Omega)\right) \ge c\, k^{-\left(\frac{1}{n}-\frac{1}{p}\right)} (1+\log k)^{-a}. \tag{15}$$

By the same duality argument as stressed above it is sufficient to show

$$a_k\left(id : B^1_{1,1}(\Omega) \longrightarrow L_{p'}(\log L)_{-a}(\Omega)\right) \;\geq\; c_1\ k^{-\left(\frac{1}{n}-\frac{1}{p}\right)}\,(1+\log k)^{-a}\ . \tag{16}$$

Note that the embedding $id^L : B^1_{1,1}(\Omega) \to L_{(p')^\sigma}(\Omega)$ can be decomposed as

$$B^1_{1,1}(\Omega) \xrightarrow{id} L_{p'}(\log L)_{-a}(\Omega) \xrightarrow{id_\sigma} L_{(p')^\sigma}(\Omega),$$

for each $\sigma > 0$ such that $(p')^\sigma > p'_0 > 1$, where p'_0 is fixed and less than p' (therefore, $0 < \sigma < \varepsilon \equiv n(\frac{1}{p'_0} - \frac{1}{p'}))$. Since from (3) it follows that $\|id_\sigma\| \leq c_1\sigma^{-a}$, with $c_1 > 0$ independent of σ, and from Proposition 4 we can write

$$a_k(id^L) \geq c_2\ k^{-(\frac{1}{n}-\frac{1}{p})}k^{-\frac{\sigma}{n}},$$

also with $c_2 > 0$ independent of σ, then the multiplicativity of the approximation numbers gives

$$a_k(id) \geq a_k(id^L)\|id_\sigma\|^{-1} \geq c_2\ c_1^{-1}\ k^{-(\frac{1}{n}-\frac{1}{p})}k^{-\frac{\sigma}{n}}\sigma^a.$$

For large k we can now choose $\sigma = an(\log k)^{-1}$ and obtain

$$a_k(id) \geq c\ k^{-(\frac{1}{n}-\frac{1}{p})}(1+\log k)^{-a},$$

with $c > 0$ independent of k. Of course this also holds for small $k \in \mathbb{N}$.
The proof is complete.

4 Related results

We collect a few closely related results and report, in particular, what was already known before.

(i) The ‘non-logarithmic’ case $a = 0$ is covered by [ET96, Thm. 3.3.4 (i), p. 119]; further improvements of this result can be found in [Cae98]. Note that the case $n = 1$, $1 < p < 2$, is not yet completely solved in this case, too.

(ii) The situation with $p = \infty$ was studied by Triebel in [Tri93]; then

$$a_k\left(id : H^{n/r}_r(\Omega) \longrightarrow L_\infty(\log L)_{-a}(\Omega)\right) \sim (1+\log k)^{-\left(a-\frac{1}{r'}\right)}, \tag{17}$$

where $1 < r < \infty$, $a > \frac{1}{r'}$, see [ET96, Thm. 3.4.2, p. 129]. Here H^s_r, $s \in \mathbb{R}$, $1 < r < \infty$, stand for the Bessel potential spaces (*fractional Sobolev spaces*).

(iii) In [EH00, Thm. 3.13] we studied the counterpart ‘lifted’ by smoothness 1, that is

$$a_k\left(id : B^{1+n/r}_{r,q}(\Omega) \longrightarrow \mathrm{Lip}^{(1,-\alpha)}(\Omega)\right) \;\sim\; (1+\log k)^{-\alpha},$$

assuming $1 < r \leq \infty$, $0 < q \leq 1$, $\alpha > 0$; here $\mathrm{Lip}^{(1,-\alpha)}(\Omega)$ are (logarithmically) refined Lipschitz spaces. Similarly we obtained in [EH00, Cor. 3.19] estimates of the type

$$a_k\left(id : \mathrm{Lip}^{(1,-\alpha)}(\Omega) \longrightarrow B^{1-s}_{\infty,q}(\Omega)\right) \;\sim\; k^{-\frac{s}{n}}\,(1+\log k)^{\alpha},$$

where $\alpha \geq 0$, $s > 0$, and $0 < q \leq \infty$, and

$$a_k\left(id : B^s_{p,q}(\Omega) \longrightarrow \mathrm{Lip}^{(1,-\alpha)}(\Omega)\right) \;\sim\; k^{-\frac{s-1}{n}+\frac{1}{p}}\,(1+\log k)^{-\alpha}$$

for $\alpha \geq 0$, $2 \leq p \leq \infty$, $0 < q \leq \infty$ and $s > 1 + \frac{n}{p}$. In the last assertion $B^s_{p,q}(\Omega)$ can be replaced by $H^s_p(\Omega)$, $2 \leq p < \infty$.

Parallel studies dealing with entropy numbers instead of approximation numbers can be found in [ET96], [EN98], [Cae00] (dealing with embeddings into spaces $L_p(\log L)_a$), and in [EH99], [EH00], [CoK01] (related to spaces $\mathrm{Lip}^{(1,-\alpha)}$).

Finally, we refer to the book [Tri01], [Har01], [Har02], and to [CaM01] for further details and results on envelopes.

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