SUBATOMIC REPRESENTATION OF BESSEL POTENTIAL SPACES MODELLED ON LORENTZ SPACES

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Dedicated to Professor J. Sampaio Martins on the occasion of his sixtieth birthday

Abstract. Subatomic representation of Bessel potential spaces modelled on Lorentz spaces are obtained via interpolation techniques.

1. Introduction

Subatomic representation for function spaces of Besov and Triebel-Lizorkin type have been considered in recent years (see [8], [9], [5], [2], for example). The proofs are quite involved, so that one can expect a lot of work if one sets as our task the use of the same type of approach in order to get subatomic representations for other function spaces. On the other hand, subatomic representations can be quite useful as a means of attack of some questions. For example, Triebel [8, 20.6, 28.6] has successfully used it in order to determine sharp upper estimates for the asymptotic behaviour of entropy numbers of embeddings between fractal-based function spaces of Besov type and, as a consequence, was able to determine the right asymptotic behaviour for the eigenvalues of some fractal pseudo-differential operators.

Our objective here is to show that with the help of interpolation techniques it is possible in some cases to take advantage of the already known subatomic representations in order to get corresponding ones for other spaces, instead of trying to overcome the many difficulties one can expect to meet if a more traditional approach is used. So, rather than trying to be as general as possible, we exemplify what we mean by setting as our task here to get subatomic representations for

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Bessel potential spaces modelled on Lorentz spaces, something which, in itself, is a new result.

Our departure point as long as subatomic representations are concerned is what is known in this respect for the usual Bessel potential spaces $H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$, $s > 0$, $1 < p < \infty$. This can be seen in [9, Section 2] or by specializing our Section 4 below for this setting (taking $q = p$ in Propositions 4.1, 4.2 and Theorem 4.3). We need to explain here what are the functions $\Psi^{\beta,\rho}_{\nu m}$ in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ which are considered there: they are the same as in [9, Corol. 2.12] and its construction depends directly only on the consideration of the dyadic resolution of unity $(\varphi_k)_{k\in\mathbb{N}_0}$ in $\mathbb{R}^n$ introduced in [9, 2.8] and, obviously, on the ρ, β, ν, m which show up in our Section 4 below. In particular, they are independent of s, p, q considered either in [9, Section 2] or in the present paper.

The Bessel potential spaces $H^sL_{pq}(\mathbb{R}^n)$, $s > 0$, $1 < p < \infty$, $1 \le q \le \infty$, for which we get subatomic representations are defined by

$$H^sL_{pq}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|((1 + |x|^2)^{s/2}\hat{f})\check{}\,\|_{L_{pq}(\mathbb{R}^n)} < \infty\},$$

where $L_{pq}(\mathbb{R}^n)$ are Lorentz spaces and $\hat{}$, $\check{}$ stand, respectively, for Fourier and inverse Fourier transformation. The subatomic results we can prove for them are collected in Section 4.

A word or two about notation: the symbol $\hookrightarrow$ shall mean that the space on the left-hand side of the symbol is continuously embedded in the space on its right-hand side; an equality sign between spaces shall mean that each one is continuously embedded in the other; the use of the same letter c in adjacent formulae does not necessarily mean they assume the same value.

2. Vector-valued Lorentz spaces

We shall consider $\mathbb{R}^n$ the usual Euclidean space endowed with Lebesgue measure.

We recall that, given a (real or complex) Banach space $(E, |\cdot|_E)$, a function $f : \mathbb{R}^n \to E$ is called strongly measurable if it is *a.e.* the limit of step functions, i.e., of functions of the form $\sum_{j=1}^N a_j\chi_{A_j}$, where a_j can be any elements of E, N can be any natural number and A_j can be any measurable subsets of $\mathbb{R}^n$ with finite measure. It is well-known (see [4, p. 124]) that, in this context, strong measurability of f is equivalent to measurability together with the hypothesis that there exists a null set Z such that $f(Z^c)$ is separable. In particular, in the case when E equals $\mathbb{R}$ or $\mathbb{C}$ the two notions of measurability coincide.

As usual (see [3, pp. 101-106]), the vector space $M(\mathbb{R}^n, E)$ of equivalence classes of strongly measurable functions from $\mathbb{R}^n$ into E (two functions being in the same class if they are equal *a.e.*, i.e., if they differ on a null set only) is endowed with the metric of convergence in measure, becoming then a topological vector space which, in particular, is Hausdorff. Also as usual, we shall sometimes refer to the elements of

$M(\mathbb{R}^n, E)$ as functions (in corresponding equivalence classes), and even functions which are only defined *a.e.* from $\mathbb{R}^n$ into E (and strongly measurable in their domains of definition) can be thought of as elements of $M(\mathbb{R}^n, E)$; the important thing is that the equivalence classes they point to in $M(\mathbb{R}^n, E)$ are clearly identified.

For $1 \le p < \infty$, the Lebesgue space $L_p(\mathbb{R}^n, E)$ is defined as the linear subspace of $M(\mathbb{R}^n, E)$ of equivalence classes containing at least a (strongly measurable) function $f : \mathbb{R}^n \to E$ such that $|f(\cdot)|_E^p$ is (Lebesgue) integrable. Endowed with

$$\|f\|_p := \Big(\int_{\mathbb{R}^n} |f(x)|_E^p \, dx \Big)^{1/p}$$

it is a Banach space continuously embedded in $M(\mathbb{R}^n, E)$. The space $L_\infty(\mathbb{R}^n, E)$ is defined as the linear subspace of $M(\mathbb{R}^n, E)$ of equivalence classes containing at least a (strongly measurable) essentially bounded function f. Endowed with $\|f\|_\infty := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|_E$, it is also a Banach space continuously embedded in $M(\mathbb{R}^n, E)$.

For any $f \in M(\mathbb{R}^n, E)$, the decreasing rearrangement of f is defined (with possibly infinite values) by

$$f^*(t) := \inf\{\sigma > 0 : \rho(f, \sigma) \le t\}, \quad t > 0,$$

where

$$\rho(f, \sigma) := |\{x \in \mathbb{R}^n : |f(x)|_E > \sigma\}|, \quad \sigma > 0.$$

Note that $f^* = |f(\cdot)|_E^*$.

Definition 2.1. *Let E be a Banach space, $1 < p < \infty$, $1 \le q < \infty$. The Lorentz space $L_{pq}(\mathbb{R}^n, E)$ is defined as*

$$L_{pq}(\mathbb{R}^n, E) := \{f \in M(\mathbb{R}^n, E) : \|f\|_{pq} := \left(\int_0^\infty (t^{1/p} f^*(t))^q \, \frac{dt}{t} \right)^{1/q} < \infty\};$$

the Lorentz space $L_{p\infty}(\mathbb{R}^n, E)$ is defined by

$$L_{p\infty}(\mathbb{R}^n, E) := \{f \in M(\mathbb{R}^n, E) : \|f\|_{p\infty} := \sup_{t > 0} t^{1/p} f^*(t) < \infty\}.$$

With the expressions $\|\cdot\|_{pq}$, these are in general only quasi-Banach spaces, but since there are norms equivalent to $\|\cdot\|_{pq}$ (see, for example, next section on interpolation), one can say that these Lorentz spaces are also Banach spaces. Moreover, it follows from properties of the decreasing rearrangement that $L_{pp}(\mathbb{R}^n, E) = L_p(\mathbb{R}^n, E)$, $1 < p < \infty$, with $\|\cdot\|_{pp} = \|\cdot\|_p$.

It is clear that if $f \in L_{pq}(\mathbb{R}^n, E)$ then also $|f(\cdot)|_E \in L_{pq}(\mathbb{R}^n, \mathbb{C})$ (or $L_{pq}(\mathbb{R}^n, \mathbb{R})$). In the sequel we shall also need a partial converse of these result:

Proposition 2.1. *Let I be a countable set and $\ell_r := \ell_r(I, \mathbb{C})$ be the Banach space of complex-valued r-summable families indexed by I, with $1 \leq r < \infty$ (see end of section for a precise definition). Let $1 < p < \infty$, $1 \leq q \leq \infty$. If*

$$\begin{array}{rccc} f: & \mathbb{R}^n & \longrightarrow & \ell_r \\ & x & \mapsto & \{f_i(x)\}_{i \in I} \end{array}$$

(possibly defined only a.e.) is such that $|f(\cdot)|_r \in L_{pq}(\mathbb{R}^n, C)$ and $f_i \in M(\mathbb{R}^n, \mathbb{C})$ for every $i \in I$, then $f \in L_{pq}(\mathbb{R}^n, \ell_r)$.

Proof. All that remains to prove is that $f \in M(\mathbb{R}^n, \ell_r)$, because then we will have $\|f\|_{pq} = \| \, |f(\cdot)|_r \, \|_{pq} < \infty$.

By hypothesis, each f_i is strongly measurable, so the same happens with the composition with the natural embedding in ℓ_r and, afterwards, with any finite sum of such compositions. Since, also by hypothesis, $(\sum_{i \in I} |f_i(\cdot)|^r)^{1/r} \in M(\mathbb{R}^n, \mathbb{C})$, so in particular the infinite sum $\sum_{i \in I} |f_i(\cdot)|^r$ is *a.e.* pointwise convergent, then f is *a.e.* the pointwise limit of the above considered finite sums, and therefore is also strongly measurable. $\square$

It is clear that the above result also holds with $\mathbb{R}$ in place of $\mathbb{C}$. However, from now on, we shall assume that all vector spaces are complex ones.

Since our function spaces will always be of "functions" defined on $\mathbb{R}^n$, from now on we shall omit $\mathbb{R}^n$ from the notation. We shall also omit the Banach space E when $E = \mathbb{C}$. Therefore, $L_{pq}(E)$ will stand for $L_{pq}(\mathbb{R}^n, E)$ and $L_{pq}(\mathbb{R}^n, \mathbb{C})$ will be simply written as L_{pq}.

We shall also need to deal with the Banach "sequence" spaces $\ell_r(I, E)$ of E-valued r-summable (if $1 \leq r < \infty$) or bounded (if $r = \infty$) families indexed by the countable set I and equipped with the norm $| \cdot |_r$:

$$\ell_r(I, E) := \Big\{ a \equiv \{a_i\}_{i \in I} \subset E : |a|_r := \Big(\sum_{i \in I} |a_i|_E^r \Big)^{1/r} < \infty \Big\},$$

where $|a|_\infty$ must be interpreted as $\sup_{i \in I} |a_i|_E$. Usually the I will be omitted from the notation if it is clear from the context. We shall also omit E when $E = \mathbb{C}$.

3. Interpolation spaces

Recall that two Banach spaces A and B are said to form an interpolation couple $\{A, B\}$ if they are both continuously embedded in the same Hausdorff topological vector space.

As far as methods of interpolation are concerned, we deal only with the real method $(\cdot, \cdot)_{\theta,q}$, where $0 < \theta < 1$ and $1 \leq q \leq \infty$ (see [7], for example).

We begin with a somewhat abstract result, dealing with the spaces of type $\ell_\infty(I, E)$ introduced before.

Proposition 3.1. *Given an interpolation couple* $\{A, B\}$ *and* $0 < \theta < 1$, $1 \leq q \leq \infty$, $\{\ell_\infty(A), \ell_\infty(B)\}$ *is also an interpolation couple and the continuous embedding*

$$(\ell_\infty(A), \ell_\infty(B))_{\theta,q} \hookrightarrow \ell_\infty((A, B)_{\theta,q})$$

holds.

The proof is straightforward, so we shall omit it.

The Lorentz spaces introduced in the preceding section can be seen as interpolation spaces of Lebesgue spaces:

Proposition 3.2. *Given a Banach space* E *and numbers* $1 < p_0, p_1 < \infty$, *with* $p_0 \neq p_1$, $1 \leq q \leq \infty$, $0 < \theta < 1$ *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, *the equality*

$$L_{pq}(E) = (L_{p_0}(E), L_{p_1}(E))_{\theta,q}$$

holds (with equivalence of (quasi-)norms).

A proof of this result in a somewhat different framework can be seen in [7, pp. 133-134]; see also [1, pp. 109-110].

Recall that $H^s L_p$ stands for H^s_p, the Bessel potential space of the tempered distributions f such that $\|f\|_{H^s L_p} := \|((1+|x|^2)^{s/2}\hat{f})\check{}\,\|_p < \infty$, $s \in \mathbb{R}$, $1 < p < \infty$. With the norm $\|\cdot\|_{H^s L_p}$, it becomes a Banach space.

Definition 3.1. *We define* $H^s L_{pq}$, *the Bessel potential space modelled on a Lorentz space, as*

$$H^s L_{pq} := \{f \in \mathcal{S}' : \|f\|_{H^s L_{pq}} := \|((1+|x|^2)^{s/2}\hat{f})\check{}\,\|_{pq} < \infty\},$$

$s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, *quasi-normed by* $\|\cdot\|_{H^s L_{pq}}$.

Though the quasi-norm above is not, in general, a norm, $H^s L_{pq}$ can still be considered as a Banach space, for an equivalent norm. This can, for example, be seen as a consequence of the next result.

Proposition 3.3. *Given numbers* $s \in \mathbb{R}$, $1 < p_0, p_1 < \infty$, *with* $p_0 \neq p_1$, $1 \leq q \leq \infty$, $0 < \theta < 1$ *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, *the equality*

$$H^s L_{pq} = (H^s L_{p_0}, H^s L_{p_1})_{\theta,q}$$

holds (with equivalence of (quasi-)norms).

Proof. We use the method of retraction and co-retraction (cf. [7, p. 22]). The operator

$$\begin{array}{rccc} S: & \mathcal{S}' & \longrightarrow & \mathcal{S}' \\ & f & \mapsto & ((1+|x|^2)^{s/2}\hat{f})\check{} \end{array}$$

is obviously a co-retraction from $H^sL_{p_i}$ into L_{p_i}, $i = 0, 1$, with S^{-1} a corresponding retraction from L_{p_i} into $H^sL_{p_i}$, $i = 0, 1$. Therefore, S also establishes a topological isomorphism from $(H^sL_{p_0}, H^sL_{p_1})_{\theta,q}$ onto the complemented subspace of $(L_{p_0}, L_{p_1})_{\theta,q}$ given by $SS^{-1}(L_{p_0}, L_{p_1})_{\theta,q}$, that is, by $(L_{p_0}, L_{p_1})_{\theta,q}$ itself, which, by the preceding proposition, equals L_{pq}. That is, $f \in \mathcal{S}'$ is in $(H^sL_{p_0}, H^sL_{p_1})_{\theta,q}$ if, and only if, $Sf \in L_{pq}$ and, moreover, $\|f\|_{(H^sL_{p_0}, H^sL_{p_1})_{\theta,q}}$ is equivalent to $\|Sf\|_{pq}$. This finishes our proof. □

Let now, for each $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, $Q_{\nu m}$ denote the cube in $\mathbb{R}^n$, with sides parallel to the coordinate axes, with centre $2^{-\nu}m$ and side length $2^{-\nu}$. Denote by $\chi_{\nu m}$ the characteristic function of $Q_{\nu m}$. Let also, from now on, I stand for $\mathbb{N}_0^n \times \mathbb{N}_0 \times \mathbb{Z}^n$.

Definition 3.2. *Given $1 < p < \infty$, define hL_p as the vector space*

$$hL_p := \Big\{\lambda \equiv \{\lambda_{(\beta,\nu,m)}\}_{(\beta,\nu,m)\in I} \subset \mathbb{C} :$$

$$\|\lambda\|_{hL_p} := \Big\| \Big(\sum_{(\beta,\nu,m)\in I} |\lambda_{(\beta,\nu,m)}\chi_{\nu m}(\cdot)|^2 \Big)^{1/2} \Big\|_p < \infty \Big\} \tag{3.1}$$

endowed with the norm $\|\cdot\|_{hL_p}$.

Note that the convergence of the sum in (3.1) is considered pointwise (*a.e.* is enough, of course). Note also that, due to Proposition 2.1, hL_p can be equivalently described as the space of all $\lambda \equiv \{\lambda_{(\beta,\nu,m)}\}_{(\beta,\nu,m)\in I} \subset \mathbb{C}$ such that $x \mapsto \{\lambda_{(\beta,\nu,m)}\chi_{\nu m}(x)\}_{(\beta,\nu,m)\in I}$ belongs to $L_p(\ell_2)$. One can take advantage of this characterization in order to get an easy proof that hL_p is a Banach space.

We would like to remark that, given $1 < p_0, p_1 < \infty$, $\{hL_{p_0}, hL_{p_1}\}$ is an interpolation couple, as both spaces are continuously embedded in (in obvious notation) the Hausdorff topological vector space $h(L_{p_0} + L_{p_1})$.

Analogously to hL_p, one can consider hL_{pq} ($1 < p < \infty$, $1 \le q \le \infty$): in the definition above one just has to use $\|\cdot\|_{pq}$ instead of $\|\cdot\|_p$ and $\|\cdot\|_{hL_{pq}}$ in place of $\|\cdot\|_{hL_p}$, though now $\|\cdot\|_{hL_{pq}}$ is generally only a quasi-norm. However, as for the big H spaces, hL_{pq} can also be considered as a Banach space, for an equivalent norm (see, for example, the proposition below).

Note that, as for hL_p, due to Proposition 2.1, hL_{pq} can be equivalently described as the space of all $\lambda \equiv \{\lambda_{(\beta,\nu,m)}\}_{(\beta,\nu,m)\in I} \subset \mathbb{C}$ such that $x \mapsto \{\lambda_{(\beta,\nu,m)}\chi_{\nu m}(x)\}_{(\beta,\nu,m)\in I}$ belongs to $L_{pq}(\ell_2)$.

Proposition 3.4. *Given numbers $1 < p_0, p_1 < \infty$, with $p_0 \ne p_1$, $1 \le q \le \infty$, $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, the equality*

$$hL_{pq} = (hL_{p_0}, hL_{p_1})_{\theta,q}$$

holds (with equivalence of (quasi-)norms).

PROOF. We use again the method of retraction and co-retraction (cf. [7, p. 22]).

We show that

$$\begin{array}{rccc} S: & hL_{p_0}+hL_{p_1} & \longrightarrow & L_{p_0}(\ell_2)+L_{p_1}(\ell_2) \\ & \{\lambda_{(\beta,\nu,m)}\}_{(\beta,\nu,m)\in I} & \mapsto & \big[x\mapsto \{\lambda_{(\beta,\nu,m)}\chi_{\nu m}(x)\}_{(\beta,\nu,m)\in I}\big] \end{array}$$

is a co-retraction from hL_{p_i} into $L_{p_i}(\ell_2)$, $i=0,1$, with

$$\begin{array}{rccc} R: & L_{p_0}(\ell_2)+L_{p_1}(\ell_2) & \longrightarrow & hL_{p_0}+hL_{p_1} \\ & \big[x\mapsto \{a_{(\beta,\nu,m)}(x)\}_{(\beta,\nu,m)\in I}\big] & \mapsto & \big\{2^{\nu n}\displaystyle\int_{\chi_{\nu m}} a_{(\beta,\nu,m)}(y)\,dy\big\}_{(\beta,\nu,m)\in I} \end{array}$$

a corresponding retraction from $L_{p_i}(\ell_2)$ into hL_{p_i}, $i=0,1$.

Using the equivalent characterization of each hL_{p_i}, given immediately after its definition, it is straightforward to see that S is well-defined, it is linear, its restriction to each hL_{p_i} is a bounded linear operator into $L_{p_i}(\ell_2)$ and, assuming R is well-defined, that the restriction of RS to each hL_{p_i} is the identity operator.

As to R, it is clear that it will be a linear well-defined operator if one can prove that its restriction to each $L_{p_i}(\ell_2)$ is well-defined taking this space into the corresponding hL_{p_i}. We show this next:

Given any $\big[x\mapsto \{a_{(\beta,\nu,m)}(x)\}_{(\beta,\nu,m)\in I}\big]\in L_{p_i}(\ell_2)$, it is clear, since p_i is assumed greater than 1, that, for each $(\beta,\nu,m)\in I$, $a_{(\beta,\nu,m)}$ is a (strongly) measurable locally integrable function, so that the expression $2^{\nu n}\int_{\chi_{\nu m}} a_{(\beta,\nu,m)}(y)\,dy$ makes sense. Furthermore, it is also clear that $x\mapsto \big|\big\{2^{\nu n}\int_{\chi_{\nu m}} a_{(\beta,\nu,m)}(y)\,dy\,\chi_{\nu m}(x)\big\}_{(\beta,\nu,m)\in I}\big|_2$ is measurable in the sense of a function with values in the extended real number system. Therefore

$$\begin{aligned} &\big\|R\big[x\mapsto \{a_{(\beta,\nu,m)}\}_{(\beta,\nu,m)\in I}\big]\big\|_{hL_{p_i}} \\ &\quad= \Big\|\Big(\sum_{(\beta,\nu,m)\in I}\Big|2^{\nu n}\int_{\chi_{\nu m}} a_{(\beta,\nu,m)}(y)\,dy\,\chi_{\nu m}(\cdot)\Big|^2\Big)^{1/2}\Big\|_{p_i} \\ &\quad\le \Big\|\Big(\sum_{(\beta,\nu,m)\in I}\Big(2^{\nu n}\int_{\chi_{\nu m}} |a_{(\beta,\nu,m)}(y)|\,dy\,\chi_{\nu m}(\cdot)\Big)^2\Big)^{1/2}\Big\|_{p_i} \\ &\quad\le \Big\|\Big(\sum_{(\beta,\nu,m)\in I}|(Ma_{(\beta,\nu,m)})(\cdot)|^2\Big)^{1/2}\Big\|_{p_i} \\ &\quad\le c\Big\|\Big(\sum_{(\beta,\nu,m)\in I}|a_{(\beta,\nu,m)}(\cdot)|^2\Big)^{1/2}\Big\|_{p_i} \\ &\quad= c\,\big\|\big[x\mapsto \{a_{(\beta,\nu,m)}(x)\}_{(\beta,\nu,m)\in I}\big]\big\|_{L_{p_i}(\ell_2)} \;<\; \infty, \end{aligned}$$

where M is the Hardy-Littlewood maximal function and we have used the maximal inequality of Fefferman-Stein (cf. [6, pp. 14-15]).

It is also now clear that the restriction of R to each $L_{p_i}(\ell_2)$ is a bounded linear operator into hL_{p_i}.

Therefore S establishes a topological isomorphism from $(hL_{p_0}, hL_{p_1})_{\theta,q}$ onto the complemented subspace of $(L_{p_0}(\ell_2), L_{p_1}(\ell_2))_{\theta,q}$ given by $SR(L_{p_0}(\ell_2), L_{p_1}(\ell_2))_{\theta,q}$, i.e., due to Proposition 3.2, onto the complemented subspace of $L_{pq}(\ell_2)$ given by $SRL_{pq}(\ell_2)$.

From this it immediately follows that

$$\left\| \left[x \mapsto \{\lambda_{(\beta,\nu,m)}\chi_{\nu m}(x)\}_{(\beta,\nu,m)\in I}\right] \right\|_{L_{pq}(\ell_2)} \leq c \left\| \{\lambda_{(\beta,\nu,m)}\}_{(\beta,\nu,m)\in I} \right\|_{(hL_{p_0}, hL_{p_1})_{\theta,q}},$$

which proves the continuous embedding

$$(hL_{p_0}, hL_{p_1})_{\theta,q} \hookrightarrow hL_{pq}.$$

On the other hand, given $\{\lambda_{(\beta,\nu,m)}\}_{(\beta,\nu,m)\in I} \in hL_{pq}$, one knows that $x \mapsto \{\lambda_{(\beta,\nu,m)}\chi_{\nu m}(x)\}_{(\beta,\nu,m)\in I}$ belongs to $L_{pq}(\ell_2)$, so that applying R one gets $\{\lambda_{(\beta,\nu,m)}\}_{(\beta,\nu,m)\in I} \in (hL_{p_0}, hL_{p_1})_{\theta,q}$ and the continuous embedding

$$hL_{pq} \hookrightarrow (hL_{p_0}, hL_{p_1})_{\theta,q}.$$

$\square$

4. Subatomic representations

Let $r \geq 0$ and $\psi \geq 0$ be a C^∞ function in $\mathbb{R}^n$ with $\operatorname{supp} \psi \subset \{y \in \mathbb{R}^n : |y| < 2^r\}$ and $\sum_{m\in\mathbb{Z}^n} \psi(x-m) = 1$ for all $x \in \mathbb{R}^n$. Given any $\beta \in \mathbb{N}_0^n$, define $\psi^\beta(x) := x^\beta \psi(x)$.

Consider, in what follows, that the real number ρ has been chosen greater than r and that $\Psi_{\nu m}^{\beta;\rho} \in \mathcal{S}$ has the same meaning as in the Introduction.

Let also I stand for $\mathbb{N}_0^n \times \mathbb{N}_0 \times \mathbb{Z}^n$ and $I' := \mathbb{N}_0 \times \mathbb{Z}^n$.

Proposition 4.1. *Let $s > 0$, $1 < p < \infty$, $1 \leq q \leq \infty$. If $f \in H^s L_{pq}$, then*

$$f = \sum_{(\beta,\nu,m)\in I} \lambda_{\nu m}^\beta(f)\, \psi^\beta(2^\nu \cdot -m), \tag{4.1}$$

summability in $\mathcal{S}'$, where the $\lambda_{\nu m}^\beta(f) := 2^{-\rho|\beta|} \left\langle f, \Psi_{\nu m}^{\beta;\rho} \right\rangle$ satisfy the relation

(4.2)
$$|\lambda(f)|_{\rho,s,p,q} := \sup_{\beta\in\mathbb{N}_0^n} 2^{\rho|\beta|} \left\| \Big(\sum_{(\nu,m)\in I'} 2^{2\nu s} |\lambda_{\nu m}^\beta(f)\chi_{\nu m}(\cdot)|^2 \Big)^{1/2} \right\|_{pq} \leq c\, \|f\|_{H^s L_{pq}},$$

for some $c > 0$ independent of the f considered.

PROOF. Choose $1 < p_0, p_1 < \infty$, with $p_0 \neq p_1$, and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and consider the linear operator

$$\begin{array}{rlcl} T: & H^s L_{p_0} + H^s L_{p_1} & \longrightarrow & \ell_\infty(L_{p_0}(\ell_2)) + \ell_\infty(L_{p_1}(\ell_2)) \quad . \\ & f & \mapsto & \left\{ x \mapsto \{2^{\nu s} \left\langle f, \Psi^{\beta,\rho}_{\nu m} \right\rangle \chi_{\nu m}(x)\}_{(\nu,m)\in I'} \right\}_{\beta \in \mathbb{N}_0^n} \end{array}$$

That this is well-defined comes from the subatomic decomposition for the spaces $H^s L_{p_i} = H^s_{p_i} = F^s_{p_i,2}$, $i = 0, 1$ — cf. [9, 2.6, 2.9, 2.11, 2.12] — together with our Proposition 2.1. Since from these references it also follows that the restriction of T to each $H^s L_{p_i}$ is bounded with values in $\ell_\infty(L_{p_i}(\ell_2))$, using real interpolation we also get that T takes $(H^s L_{p_0}, H^s L_{p_1})_{\theta,q}$ linearly and boundedly into $(\ell_\infty(L_{p_0}(\ell_2)), \ell_\infty(L_{p_1}(\ell_2)))_{\theta,q}$. Now just conjugate this with Propositions 3.1, 3.2 and 3.3 in order to get (4.2). As to (4.1), it follows immediately from the corresponding result for the elements of $H^s L_{p_i}$, $i = 0, 1$ — cf. [9, 2.6, 2.9, 2.11, 2.12] — and the fact that each $f \in H^s L_{pq}$ can be written as $f = f_0 + f_1$, with $f_0 \in H^s L_{p_0}$, $f_1 \in H^s L_{p_1}$. $\square$

Proposition 4.2. *Let $s > 0$, $1 < p < \infty$, $1 \leq q \leq \infty$. If $\lambda \equiv \{\lambda^\beta_{\nu m}\}_{(\beta,\nu,m)\in I} \subset \mathbb{C}$ is such that*

$$|\lambda|_{\rho,s,p,q} := \sup_{\beta \in \mathbb{N}_0^n} 2^{\rho|\beta|} \Big\| \Big(\sum_{(\nu,m)\in I'} 2^{2\nu s} |\lambda^\beta_{\nu m} \chi_{\nu m}(\cdot)|^2 \Big)^{1/2} \Big\|_{pq} < \infty, \tag{4.3}$$

then

$$f := \sum_{(\beta,\nu,m)\in I} \lambda^\beta_{\nu m}\, \psi^\beta(2^\nu \cdot - m) \tag{4.4}$$

(summability in $\mathcal{S}'$) belongs to $H^s L_{pq}$ and there is some constant $c > 0$ (independent of f and λ) such that

$$\|f\|_{H^s L_{pq}} \leq c \inf |\lambda|_{\rho,s,p,q}, \tag{4.5}$$

where the infimum runs over all λ satisfying (4.3) and giving rise to the same f according to (4.4).

PROOF. Given λ according to the hypotheses and $\varepsilon > 0$, using summability properties, properties of the decreasing rearrangement of functions and the fact that L_{pq} can be viewed as a Banach space continuously embedded in $L_{p_0} + L_{p_1}$, for some

$1 < p_0, p_1 < \infty$, we have that

$$\Big\| \Big(\sum_{(\beta,\nu,m)\in I} 2^{2(\rho-\varepsilon)|\beta|}\, 2^{2\nu s} |\lambda^\beta_{\nu m} \chi_{\nu m}(\cdot)|^2 \Big)^{1/2} \Big\|_{pq}$$

$$\leq \Big\| \sum_{\beta\in\mathbb{N}_0^n} 2^{-\varepsilon|\beta|}\, 2^{\rho|\beta|} \Big(\sum_{(\nu,m)\in I'} 2^{2\nu s} |\lambda^\beta_{\nu m} \chi_{\nu m}(\cdot)|^2 \Big)^{1/2} \Big\|_{pq}$$

$$\leq c \sum_{\beta\in\mathbb{N}_0^n} 2^{-\varepsilon|\beta|}\, 2^{\rho|\beta|} \Big\| \Big(\sum_{(\nu,m)\in I'} 2^{2\nu s} |\lambda^\beta_{\nu m} \chi_{\nu m}(\cdot)|^2 \Big)^{1/2} \Big\|_{pq}$$

$$\leq c \Big(\sum_{\beta\in\mathbb{N}_0^n} 2^{-\varepsilon|\beta|} \Big) |\lambda|_{\rho,s,p,q} \ < \ \infty .$$

Assume, from now on, that $\varepsilon > 0$ has been chosen in such a way that $\rho - \varepsilon > r$.

Choose $1 < p_0, p_1 < \infty$, with $p_0 \neq p_1$, and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and consider the linear operator

$$\begin{array}{llll} U: & hL_{p_0} + hL_{p_1} & \longrightarrow & H^s L_{p_0} + H^s L_{p_1} \quad , \\ & \{\lambda^\beta_{\nu m}\}_{(\beta,\nu,m)\in I} & \mapsto & \displaystyle\sum_{(\beta,\nu,m)\in I} 2^{-(\rho-\varepsilon)|\beta|}\, 2^{-\nu s} \lambda^\beta_{\nu m}\, \psi^\beta(2^\nu \cdot - m) \end{array}$$

where the convergence (summability) of the sum is considered in $\mathcal{S}'$ (or in $L_{p_0} + L_{p_1}$, if one wishes some more precise information). That this is well-defined follows easily from the subatomic representation for the spaces $H^s L_{p_i} = H^s_{p_i} = F^s_{p_i,2}$, $i = 0,1$ — cf. [9, 2.6, 2.7, 2.9]. Since these references also guarantee that the restriction of U to each hL_{p_i} is bounded with values in $H^s L_{p_i}$, using real interpolation one gets that U takes $(hL_{p_0}, hL_{p_1})_{\theta,q}$ linearly and boundedly into $(H^s L_{p_0}, H^s L_{p_1})_{\theta,q}$. Conjugating this with Propositions 3.3 and 3.4 and the first part of the present proof, it follows that f given by (4.4) is well-defined (with summability meant in $\mathcal{S}'$, or even in $L_{p_0} + L_{p_1}$) and (4.5) holds true. □

Remark 4.1. *We would like to stress that the summability implied by (4.4) in $\mathcal{S}'$ is not an assumption, but rather a consequence of (4.3). Also that the summability can even be taken in $L_{p_0} + L_{p_1}$, for suitable $1 < p_0, p_1 < \infty$ according to the proof given above.*

Now the theorem on the subatomic representation for spaces $H^s L_{pq}$ which follows is a simple corollary of the two preceding propositions:

Theorem 4.3. *Let $s > 0$, $1 < p < \infty$, $1 \leq q \leq \infty$. Then $f \in \mathcal{S}'$ belongs to $H^s L_{pq}$ if, and only if, it can be represented by*

$$f = \sum_{(\beta,\nu,m)\in I} \lambda^\beta_{\nu m}\, \psi^\beta(2^\nu \cdot - m) \tag{4.6}$$

(summability in $\mathcal{S}'$) for some $\lambda \equiv \{\lambda^\beta_{\nu m}\}_{(\beta,\nu,m)\in I} \subset \mathbb{C}$ *satisfying* $|\lambda|_{\rho,s,p,q} < \infty$, *where* $|\cdot|_{\rho,s,p,q}$ *has the same meaning as in (4.3).*

Moreover, an equivalent quasi-norm in H^sL_{pq} *is given by*

$$\inf |\lambda|_{\rho,s,p,q},$$

where, for each $f \in H^sL_{pq}$, *the infimum is taken over all families* $\lambda \equiv \{\lambda^\beta_{\nu m}\}_{(\beta,\nu,m)\in I} \subset \mathbb{C}$ *satisfying (4.6) and such that* $|\lambda|_{\rho,s,p,q}$ *is finite.*

Further, given any $f \in H^sL_{pq}$, *one can choose* λ *in an optimal way (in the sense that — besides (4.6) — it verifies*

$$|\lambda|_{\rho,s,p,q} \le c\,\|f\|_{H^sL_{pq}},$$

for some $c > 0$ *independent of* f*), namely*

$$\lambda = \lambda(f) = \{2^{-\rho|\beta|}\left\langle f, \Psi^{\beta,\rho}_{\nu m}\right\rangle\}_{(\beta,\nu,m)\in I}.$$

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