About approximation numbers in function spaces

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Abstract

Sharp estimates for the approximation numbers of embeddings between the function spaces B^s_{pq} and F^s_{pq} on domains are given in a case not thoroughly studied by Edmunds and Triebel. Corresponding sharp estimates are also obtained for the counterparts of that case in the weighted function space setting.

1 Introduction

Approximation numbers of embeddings between function spaces have been studied in recent years in the general framework of the scales of spaces B^s_{pq} and F^s_{pq} on domains [4], [5]. More recently, the weighted counterparts of those embeddings have also been dealt with [7]. The estimates (upper and lower) for the approximation numbers depend on the relationship between the parameters involved, and the same happens with the quality of the estimates: in some cases we have sharp ones; in other cases we don't. One refers to the latter cases as being critical: the relationship between the parameters is such that the technique used then fails.

Our point here is that there are some cases for which sharp estimates have been overlooked, though they do not really fall into the category of a critical situation. Our aim is to give the correct picture in these cases.

Let's describe the problem in a schematic way: when considering the embedding $B^{s_1}_{p_1q_1}(\Omega) \to B^{s_2}_{p_2q_2}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $s_1, s_2 \in \mathbb{R}$, $p_1, p_2, q_1, q_2 \in]0, \infty]$ and $\delta^+ \equiv s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0$, Edmunds and Triebel in [5] left out, for example, the case $0 < p_1 < 2 < p_2 < \infty$ when $s_1 - s_2 \leq n \max\{1 - 1/p_2, 1/p_1\}$. They say that the question of the true rate of decay of the approximation numbers of the embedding in this case remains open and also point out that it is not known whether there is such a rate in this case or not. This is, however, in contrast with what König writes in [10, 3.c.7(1)], from which it seems that, at least in the classical framework of Sobolev and Besov spaces, a true rate of decay exists and is known for that case (except for critical relationships of the parameters). In fact, in this context it has surely been known for some time, as from the estimates for the approximation numbers of embeddings between sequence spaces, made available by Gluskin [6], it is clear how to use the discretization technique of Maiorov [13] to get the result. Such an approach was, for example, used in the work of Lubitz [12] in order to get the true rate of decay for Kolmogorov and Weyl numbers of classical Sobolev embeddings, this time taking advantage of estimates for the same type of numbers in sequence spaces. Moreover, in these estimates for the Weyl and Kolmogorov numbers, as well as for the approximation numbers dealt with below, the phenomenon (already noticed by Kashin [9] in a similar context — see also [11]) of the change of asymptotics for small smoothness shows up. Unfortunately, it is not easy, at least in the West, to find a reference for those results concerning the approximation numbers of classical Sobolev-Besov embeddings. As a consequence, we can see, for example, that in [7] some cases in the study of the approximation numbers of embeddings between weighted function spaces could not be satisfactorily dealt with because the author of that paper was not aware of sharp estimates in the aforementioned case.

In view of this, the present work also aims to put an end to this state of affairs.

The plan of the paper is as follows. In Section 2 we collect, and prove as necessary, the relevant results for approximation numbers of identity maps between sequence spaces. In Section 3 we consider the case, mentioned above, left out in [5] and show that for the non-critical relationship $s_1 - s_2 < n \max\{1 - 1/p_2, 1/p_1\}$ one gets

$$k^{-\frac{\delta^+}{2n}\min\{p_1',p_2\}}$$

as the true rate of decay of the approximation numbers a_k (and we also give some complements there). We would like to stress that this is not just a proof of a known (though not widely publicized) result: the scales of spaces B^s_{pq} and F^s_{pq} we deal with include a variety of classical spaces, but also a variety of other spaces (for details, refer to [15]); in particular, the parameters p and q are allowed to be positive numbers less than 1, in which case, instead of Banach function spaces, we are dealing with quasi-Banach function spaces. In Section 4 we deal with the influence of the preceding estimate in the context of weighted function spaces, so that we are able to improve the results of [7] (see Subsection 4.3 below for a summary of what can be said as a result of our study).

We recall here the definition of the k-th approximation number $a_k(T)$ (with $k \in \mathbb{N}$) of the continuous linear operator $T : B_1 \to B_2$, where B_1 and B_2 are two quasi-Banach spaces:

$$a_k(T) \equiv \inf_S \sup\{\|Tu - Su|B_2\| : \|u|B_1\| \leq 1\},$$

where the infimum is taken over all continuous linear operators $S : B_1 \to B_2$ such that rank $S < k$. Here $\|.|B\|$ denotes the quasi-norm in the quasi-normed space B, though we shall write simply $|.|$ for the Euclidean norm in $\mathbb{R}^n$.

We shall occasionally need to refer to $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, which stand, respectively, for the space of complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ equipped with the usual topology and for the space of tempered distributions equipped with the strong topology. We will use then the notation $^\wedge$ and $^\vee$ to denote the Fourier transformation and its inverse, respectively.

Finally, positive constants the precise values of which have no influence on the estimates will be just denoted by c, occasionally with additional subscripts to distinguish between them within the same formula or the same step of a proof.

2 Required results in sequence spaces

Let $m \in \mathbb{N}$, $p \in]0, \infty]$ and ℓ^m_p be the linear space of all complex m-tuples $y \equiv (y_i)^n_{i=1}$ furnished with the quasi-norm

$$\|y|\ell^m_p\| \equiv \left(\sum_{j=1}^m |y_j|^p\right)^{1/p}$$

(usual modification if $p = \infty$). Define p' by $\frac{1}{p} + \frac{1}{p'} = 1$ if $p \in [1, \infty]$ and by $p' = \infty$ if $p \in]0, 1[$. Let a^m_k be the k-th approximation number of the natural embedding

$$id^m : \ell^m_{p_1} \to \ell^m_{p_2},$$

where $p_1, p_2 \in]0, \infty]$.

We have the following results (mainly due to Gluskin [6], though the extension to $p_1 < 1$ is being taken from [5, 3.2.2]), where $a_k^m \approx K$ means that a_k^m/K is bounded above and below by positive constants independent of m and k.

Lemma 2.1 *(i) Let $1 \leq p_1 \leq 2 \leq p_1' \leq p_2 \leq \infty$, with $(p_1, p_2) \neq (1, \infty)$ and $k \leq m/2$. Then*

$$a_k^m \approx \min\{1, m^{1/p_1'} k^{-1/2}\}.$$

(ii) Let $0 < p_1 < 2 \leq p_2 < p_1'$ and $k \leq m/2$. Then

$$a_k^m \approx \min\{1, m^{1/p_2} k^{-1/2}\}.$$

From these one easily gets the following estimates.

Corollary 2.2 *Let $0 < p_1 \leq 2 \leq p_2 < \infty$ (or $1 < p_1 \leq 2 < p_2 = \infty$). Then*

(i) there is $c > 0$ such that, for all $k, m \in \mathbb{N}$,

$$a_k^m \leq c m^{1/\min\{p_1', p_2\}} k^{-1/2};$$

(ii) there is $c > 0$ such that, for all $k, m \in \mathbb{N}$ with $k \leq \frac{1}{2} m^{2/\min\{p_1', p_2\}}$,

$$a_k^m \geq c.$$

Proof. (i) Consider the composition

$$\ell_{p_1}^m \xrightarrow{J} \ell_{p_1}^{2m} \xrightarrow{id^{2m}} \ell_{p_2}^{2m} \xrightarrow{P} \ell_{p_2}^m,$$

where $J(\xi_i)_{i=1}^m \equiv (\xi_1, \ldots, \xi_m, 0, \ldots, 0)$ and $P(\xi_i)_{i=1}^{2m} \equiv (\xi_i)_{i=1}^m$, apply the Lemma to id^{2m} and use the multiplicativity of the approximation numbers to get, for $k \leq m$,

$$a_k^m \leq c(2m)^{1/\min\{p_1', p_2\}} k^{-1/2}.$$

The required result then follows by redefining the constant c and taking into account the well-known fact that $a_k^m = 0$ when $k > m$.

(ii) Note that $\min\{p_1', p_2\} \geq 2$, so that the assumption $k \leq \frac{1}{2} m^{2/\min\{p_1', p_2\}}$ implies that $k \leq m/2$ and $m^{1/\min\{p_1', p_2\}} k^{-1/2} \geq \sqrt{2} > 1$. We again obtain the stated result by applying the Lemma.

3 Approximation numbers in unweighted function spaces

Let Ω be a non-empty bounded open subset of $\mathbb{R}^n$ with C^∞ boundary $\partial\Omega$. Let $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$, for $s \in \mathbb{R}$ and $p, q \in]0, \infty]$ ($p \in]0, \infty[$ in the F-case), be the function spaces extensively studied in the books [15] and [16] of Triebel — to which we refer for definitions and properties (we just note that these scales of spaces include the classical Sobolev and Besov spaces defined on Ω).

Denote by a^{BB}_k the k-th approximation number of the natural embedding $B^{s_1}_{p_1q_1}(\Omega) \to B^{s_2}_{p_2q_2}(\Omega)$, where $s_1, s_2 \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in]0, \infty]$ are such that $\delta^+ \equiv s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0$. We also use the self-explanatory notation a^{BF}_k, a^{FB}_k and a^{FF}_k to cover all possibilities of B- and F-spaces in the domain and the target spaces of the embedding.

In [4], [5], Edmunds and Triebel studied these numbers, obtaining sharp estimates (in the sense of $\approx$ with a constant independent of k) except in the following cases:

(i) $0 < p_1 \leq 1 < p_2 = \infty$;

(ii) $0 < p_1 < 2 < p_2 < \infty$ (or $1 < p_1 < 2 < p_2 = \infty$) and $s_1 - s_2 \leq n \max\{1 - 1/p_2, 1/p_1\}$.

The reason for the exception in case (i) has to do with the lack of corresponding precise estimates in sequence spaces, and we have nothing further to add here. However in case (ii), apart from the critical situation when $s_1 - s_2 = n \max\{1 - 1/p_2, 1/p_1\}$, it is possible to get sharp estimates in a streamlined way, by using what is known for the corresponding situations in sequence spaces.

Before proceeding we would like to remark that $s_1 - s_2 < n \max\{1 - 1/p_2, 1/p_1\}$ if and only if $\delta^+ < n/\min\{p_1', p_2\}$, under the assumption $p_1 \leq p_2$. Note also that in this section it will always be $\delta^+ = s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, since we will always have $p_1 \leq p_2$. For future reference it is convenient to define $\delta \equiv s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$ and remark that $s_1 - s_2 < n \max\{1 - 1/p_2, 1/p_1\}$ if and only if $\delta < n/\min\{p_1', p_2\}$ (irrespective of the order relation between p_1 and p_2). Of course, $\delta = \delta^+$ if $p_1 \leq p_2$.

Theorem 3.1 *Let $s_1, s_2 \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in]0, \infty]$ be such that $\delta^+ > 0$. Let $0 < p_1 < 2 < p_2 < \infty$ (or $1 < p_1 < 2 < p_2 = \infty$) and $\delta < n/\min\{p_1', p_2\}$. Then*

$$a^{BB}_k \approx k^{-\frac{\delta}{2n}\min\{p_1', p_2\}}$$

Proof. (i) **Upper estimate**

The idea is to use the discretization technique given in [1, II.4.8] — see also [3, Prop. 2.2.3] — and the first part of the proof of Theorem II.4.9 in [1] — see also [2, 3.3.2] —, so that

$$(a^{BB}_{c_1k})^\rho \leq c_2 \left(2^{-N\delta\rho} + \sum_{j=L}^{N} 2^{-j\delta\rho}(a^{M_j}_{r_j})^\rho\right), \tag{1}$$

where c_1, c_2 are positive constants (i.e., positive numbers independent of k), $\rho \equiv \min\{1, p_2, q_2\}$, $L = [\frac{1}{n}\log_2 k]$, $N = [\frac{\gamma}{n}\log_2 k]$, $\gamma(\geq 1)$ is to be fixed later on independently of k, M_j is the number of $m \in \mathbb{Z}^n$ such that $|m| \leq 2^{j+2}\sqrt{n}$, $r_j = [k^{1-\gamma\varepsilon/n}2^{j\varepsilon}]$ and $\varepsilon(>0)$ is to be fixed later on independently of k and as small as we wish (in order that the inequality $r_j \geq 1$ holds true). Note that a positive constant c_3 can be found such that $M_j \leq c_3 2^{jn}$.

The use of Corollary 2.2(i) within the summation $\sum_{j=L}^N$ leads to the inequalities

$$\begin{aligned}\sum_{j=L}^N 2^{-j\delta\rho}(a_{r_j}^{M_j})^\rho &\leq \sum_{j=L}^N 2^{-j\delta\rho} c_4 M_j^{\rho/\min\{p_1',p_2\}}[k^{1-\gamma\varepsilon/n}2^{j\varepsilon}]^{-\rho/2} \\ &\leq c_5 k^{-\rho/2+\gamma\varepsilon\rho/(2n)} \sum_{j=L}^N 2^{j\rho(-\delta+n/\min\{p_1',p_2\}-\varepsilon/2)}.\end{aligned}$$

The hypothesis $\delta < n/\min\{p_1', p_2\}$ permits us to choose $\varepsilon > 0$ in such a way that $-\delta + n/\min\{p_1', p_2\} - \varepsilon/2 > 0$ and so

$$\sum_{j=L}^N 2^{-j\delta\rho}(a_{r_j}^{M_j})^\rho \leq c_6 k^{-\rho/2-\gamma\rho\delta/n+\gamma\rho/\min\{p_1',p_2\}}. \tag{2}$$

Comparing this with the term $2^{-N\delta\rho}$ of (1), which is $O(k^{-\gamma\rho\delta/n})$ as $k \to \infty$, we see that for optimal results one should choose γ in such a way that $-\rho/2 + \gamma\rho/\min\{p_1', p_2\} = 0$, that is, $\gamma = \min\{p_1', p_2\}/2$; since this is greater than 1, it is a possible choice, so that putting (2) in (1) gives

$$(a_{c_1 k}^{BB})^\rho \leq c_7 k^{-\gamma\rho\delta/n} = c_7 k^{-\delta\min\{p_1',p_2\}\rho/(2n)},$$

from which the stated upper estimate follows .

(ii) **Lower estimate**

We use the fact that there is $c_1 > 0$ such that, for all $j, k \in \mathbb{N}$,

$$a_k^{BB} \geq c_1 2^{-j\delta} a_k^{N_j}, \tag{3}$$

with $N_j = 2^{jn}$ (cf. [5, 4.3.1]).

For each $k \in \mathbb{N}$ we choose $j \in \mathbb{N}$ such that

$$\frac{1}{2}2^{(j-1)2n/\min\{p_1',p_2\}} \leq k \leq \frac{1}{2}2^{j2n/\min\{p_1',p_2\}}.$$

Using part (ii) of Corollary 2.2 in (3) we obtain the inequalities

$$a_k^{BB} \geq c_2 2^{-j\delta} = c_2 2^{-\delta}\left(2^{(j-1)2n/\min\{p_1',p_2\}}\right)^{-\delta\min\{p_1',p_2\}/(2n)} \geq c_3 k^{-\delta\min\{p_1',p_2\}/(2n)},$$

and the proof is complete.

Remark. We didn't need the hypothesis $\delta < n/\min\{p_1', p_2\}$ to prove the lower estimate.

Corollary 3.2 *Under the same hypotheses of the preceding theorem (except that when F-spaces are involved the corresponding parameter p must not be ∞), the result holds true for any of a_k^{BF}, a_k^{FB} or a_k^{FF} instead of a_k^{BB}.*

Proof. This follows as in [5, 2.1.5].

4 Approximation numbers in weighted function spaces

Let $B^s_{pq}(\alpha) \equiv B^s_{pq}((1+|x|^2)^{\alpha/2})$ and $F^s_{pq}(\alpha) \equiv F^s_{pq}((1+|x|^2)^{\alpha/2})$, for $\alpha, s \in \mathbb{R}$ and $p, q \in]0,\infty]$ ($p \in]0,\infty[$ in the F-case), be weighted function spaces (with $(1+|x|^2)^{\alpha/2}$ the weight function) corresponding to $B^s_{pq} \equiv B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq} \equiv F^s_{pq}(\mathbb{R}^n)$, respectively (for definitions and properties, see [7] and references therein).

Denote by a_k^B the k-th approximation number of the natural embedding $B^{s_1}_{p_1q_1}(\alpha) \to B^{s_2}_{p_2q_2}$, where $\alpha, s_1, s_2 \in \mathbb{R}$, $p_1, p_2 \in]0,\infty[$ and $q_1, q_2 \in]0,\infty]$ are such that $s_1 > s_2$, $\alpha > n\left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+$ and $\delta \equiv s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > 0$. Analogously, a_k^F will stand for the k-th approximation number of the natural embedding $F^{s_1}_{p_1q_1}(\alpha) \to F^{s_2}_{p_2q_2}$, with the same restrictions on the parameters.

In [7] Haroske studied these numbers off the critical line $\delta = \alpha$, obtaining sharp estimates (in the sense of $\approx$ with a constant independent of k) except in the following four cases:

(i) $0 < p_1 < 2 < p_2 < \infty$ and $\delta > \alpha > n/\min\{p_1', p_2\}$;

(ii) $0 < p_1 < 2 < p_2 < \infty$, $\alpha > \delta$ and $\delta \leq n/\min\{p_1', p_2\}$;

(iii) $0 < p_1 < 2 < p_2 < \infty$ and $n/\min\{p_1', p_2\} \geq \delta > \alpha$;

(iv) $0 < p_1 < 2 < p_2 < \infty$ and $\delta > n/\min\{p_1', p_2\} \geq \alpha$.

These correspond, respectively, to what Haroske calls regions IV, VII, VIII and IX in [7, 3.2].

We have not much to say about case (i): it is possible to reduce it to the study of what happens on the critical line $\delta = \alpha$, but then the best we can do is to show that the power exponent of k for the lower estimate given by Haroske, namely $-\alpha/n - \min\{1/p_1 - 1/2, 1/2 - 1/p_2\}$, is the correct exponent for the upper estimate, even if we can't get rid of a perturbing factor of the type of a positive power of $\log(1+k)$. We will not dwell upon this here, as it is more or less clear from the results in [7].

As in the context of unweighted function spaces, we would like to get sharp estimates for the remaining cases in a streamlined way. This will necessarily rule out some critical situations, but there might be some surprises here. We shall come back to this again later on.

4.1 Lower estimates

4.1.1 In case (ii)

$$a_k^B \geq c k^{-\frac{\delta}{2n}\min\{p_1', p_2\}},$$

for some $c > 0$ independent of k.

In fact, as in [7, 4.2, Step 1], we can write $a_k^B \geq c a_k^{BB}$ (recall the notation used in our Section 3 — in particular, the Ω we are considering can be any fixed non-empty bounded open subset of $\mathbb{R}^n$ with C^∞ boundary), so that we get the stated result by applying Theorem 3.1 and the Remark that follows it.

Observe also that, by standard arguments (cf. [7, 3.2]), the same lower estimate holds for a_k^F.

4.1.2 In cases (iii) and (iv)

$$a_k^F \geq c k^{-\frac{\alpha}{2n}\min\{p_1', p_2\}},$$

for some $c > 0$ independent of k.

Actually, since this estimate does not depend on the parameters s and q and we are assuming $\delta > \alpha$, an argument as in [7, 4.2, Step 2] shows that the same estimate holds also for a_k^B. Accordingly, we shall concentrate here on proving it for a_k^F only.

We use [7, (4.2/17)], namely that there is $c_1 > 0$ such that, for all $j, k \in \mathbb{N}$,

$$a_k^F \geq c_1 2^{-j\alpha} a_k^{N_j} \tag{4}$$

with $N_j = 2^{jn}$, and proceed as in the context of the unweighted function spaces: for each $k \in \mathbb{N}$ we choose $j \in \mathbb{N}$ such that

$$\frac{1}{2} 2^{(j-1)2n/\min\{p_1', p_2\}} \leq k \leq \frac{1}{2} 2^{j2n/\min\{p_1', p_2\}}$$

and use part (ii) of Corollary 2.2 in (4) to conclude that

$$a_k^F \geq c_2 2^{-j\alpha} \geq c_3 k^{-\frac{\alpha}{2n}\min\{p_1', p_2\}}.$$

4.2 Upper estimates

4.2.1 A localization technique

For each $j \in \mathbb{N}$ consider the operators

$$F_j : B^{s_1}_{p_1 q_1}(\alpha) \to B^{s_2}_{p_2 q_2} \text{ given by } F_j f = \varphi_j f,$$

where $(\varphi_j)_{j\in\mathbb{N}}$ is a dyadic resolution of unity defined in the following way: $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ is chosen so that supp $\varphi_0 \subset \{x \in \mathbb{R}^n : |x| < 2\}$ and $\varphi_0(x) = 1$ if $|x| \leq 1$; $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)$ for each $j \in \mathbb{N}$.

Define also, for given $L \in \mathbb{N}$, $F^L = id^B - \sum_{j=0}^L F_j$, where id^B is the natural embedding $B^{s_1}_{p_1 q_1}(\alpha) \to B^{s_2}_{p_2 q_2}$. We remark that $F^L f = (1 - \varphi(2^{-N}\cdot))f$ for every $f \in B^{s_1}_{p_1 q_1}(\alpha)$.

Proposition 4.1 *Assume* $s_1 > n(1/p_1 - 1)_+$ *and* $s_2 < 0$*, together with the general conditions on* $\alpha, s_1, s_2, p_1, p_2, q_1$ *and* q_2 *set forth at the beginning of Section 4. Let* $\rho \equiv \min\{1, p_2, q_2\}$*,* $k, L \in \mathbb{N}$*. Let* $k_j \in \mathbb{N}$*,* $j \in \{0, \ldots, L\}$*, be such that* $k = \sum_{j=0}^{L} k_j$*. There is a positive constant* c *(independent of* k, L, j *and the* k_j*) such that*

$$(a_k^B)^\rho \leq c \left(2^{-L\alpha\rho} + \sum_{j=0}^{L} 2^{j(\delta-\alpha)\rho} (a_{k_j}^{BB})^\rho \right),$$

where the Ω *involved in* $a_{k_j}^{BB}$ *is here the set* $\{x \in \mathbb{R}^n : |x| < 2\}$*.*

Proof. The result follows by the same reasoning as in [8, pp. 151-152] for the entropy numbers, where homogeneity arguments were used.

We shall want to apply this localization technique to the cases (ii) and (iii) mentioned before in this Section 4 and take advantage of the already known estimates for $a_{k_j}^{BB}$ in these situations. As we have seen in Section 3, if we make the further assumption that δ be strictly less than $n/\min\{p_1', p_2\}$ in these two cases, then we can write

$$a_k^{BB} = O(k^{-\frac{\delta}{2n}\min\{p_1', p_2\}}) \quad \text{as} \quad k \to \infty.$$

If we also assume that $s_1 > n(1/p_1 - 1)_+$ and $s_2 < 0$, then we can apply the preceding proposition and obtain the inequality

$$(a_k^B)^\rho \leq c \left(2^{-L\alpha\rho} + \sum_{j=0}^{L} 2^{j(\delta-\alpha)\rho} k_j^{-\delta\rho\min\{p_1', p_2\}/(2n)} \right), \tag{5}$$

where the meaning of the letters is as in Proposition 4.1.

At this point we can get rid of the annoying restrictions $s_1 > n(1/p_1 - 1)_+$ and $s_2 < 0$. To that effect, we use the fact that the lift operator I_σ on $\mathcal{S}'(\mathbb{R}^n)$ (for $\sigma \in \mathbb{R}$), given by

$$I_\sigma f = ((1 + |x|^2)^{\sigma/2} \hat{f})^\vee,$$

maps B_{pq}^s isomorphically onto $B_{pq}^{s-\sigma}$ and also $B_{pq}^s(\alpha)$ isomorphically onto $B_{pq}^{s-\sigma}(\alpha)$ (see [15, 2.3.8] and [14, Chapter 5] and the references given there). We proceed then as follows.

Let $\alpha, s_1, s_2, p_1, p_2, q_1, q_2$ be required to satisfy only the inequalities $0 < p_1 < 2 < p_2 < \infty$, together with the general conditions set forth at the beginning of Section 4. Consider s_0 such that $s_1 > s_0 > s_2$ and $s_1' \equiv s_1 - s_0 > n(1/p_1 - 1)_+$ and $s_2' \equiv s_2 - s_0 < 0$. Then (5) holds for the embedding $B_{p_1q_1}^{s_1'}(\alpha) \to B_{p_2q_2}^{s_2'}$. If we now write the embedding $B_{p_1q_1}^{s_1}(\alpha) \to B_{p_2q_2}^{s_2}$ as the composition

$$B_{p_1q_1}^{s_1}(\alpha) \xrightarrow{I_{s_0}} B_{p_1q_1}^{s_1-s_0}(\alpha) \to B_{p_2q_2}^{s_2-s_0} \xrightarrow{I_{s_0}^{-1}} B_{p_2q_2}^{s_2},$$

apply the multiplicativity of the approximation numbers and use the fact that $s_1' - s_2' - n(1/p_1 - 1/p_2) = s_1 - s_2 - n(1/p_1 - 1/p_2) = \delta$, we obtain the inequality (5) without the restrictions made for s_1 and s_2 in Proposition 4.1.

For future reference, we state carefully the result we have just proved.

Corollary 4.2 *Assume $0 < p_1 < 2 < p_2 < \infty$ and $\delta < n/\min\{p_1', p_2\}$, together with the general conditions on $\alpha, s_1, s_2, p_1, p_2, q_1$ and q_2 set forth at the beginning of Section 4. Let ρ, k, L and the k_j be as in the Proposition. Then there is a positive constant c (independent of k, L, j and the k_j) such that (5) holds true.*

4.2.2 The case (ii)

In the case (ii) under the further restriction $\delta < n/\min\{p_1', p_2\}$ we obtain the inequality

$$a_k^B \leq ck^{-\frac{\delta}{2n}\min\{p_1', p_2\}},$$

for some $c > 0$ independent of k.

In fact, use Corollary 4.2 with $k_j = [k2^{-\varepsilon j} + 1]$, $j = 0, \ldots, L$, and $L = [\frac{\gamma}{n}\log_2 k + 1]$, where $\varepsilon, \gamma (> 0)$ are to be fixed later on independently of k. Note that $\sum_{j=0}^{L} k_j \leq c_1 k$, so that

$$(a_{c_1 k}^B)^\rho \leq c(k^{-\alpha\gamma\rho/n} + k^{-\delta\rho\min\{p_1', p_2\}/(2n)}),$$

where $\varepsilon > 0$ was chosen in such a way that $\alpha - \delta - \varepsilon\delta\min\{p_1', p_2\}/(2n) > 0$. If we choose now $\gamma = \delta\min\{p_1', p_2\}/(2\alpha)$ we obtain the inequality

$$a_{c_1 k}^B \leq ck^{-\delta\min\{p_1', p_2\}/(2n)},$$

and, clearly, the same estimate holds if we substitute a_k^B for $a_{c_1 k}^B$.

4.2.3 The case (iii)

In the case (iii) under the further restriction $\delta < n/\min\{p_1', p_2\}$ we obtain the inequality

$$a_k^B \leq ck^{-\frac{\alpha}{2n}\min\{p_1', p_2\}},$$

for some $c > 0$ independent of k.

In fact, use Corollary 4.2 with $k_j = [k^{1-\varepsilon\gamma/n}2^{\varepsilon j} + 1]$, $j = 0, \ldots, L$, and $L = [\frac{\gamma}{n}\log_2 k + 1]$, where $\varepsilon, \gamma (> 0)$ are to be fixed later on independently of k. Note that $\sum_{j=0}^{L} k_j \leq c_1 k$, so that

$$(a_{c_1 k}^B)^\rho \leq c(k^{-\alpha\gamma\rho/n} + k^{-\alpha\gamma\rho/n + \delta\gamma\rho/n - \delta\rho\min\{p_1', p_2\}/(2n)}),$$

where $\varepsilon > 0$ was chosen in such a way that $\delta - \alpha - \varepsilon\delta\min\{p_1', p_2\}/(2n) > 0$. If we choose now $\gamma = \min\{p_1', p_2\}/2$ we obtain the inequality

$$a_{c_1 k}^B \leq ck^{-\alpha\min\{p_1', p_2\}/(2n)},$$

and, clearly, the same estimate holds if we substitute a_k^B for $a_{c_1 k}^B$.

4.2.4 The case (iv)

We can now study the case (iv) under the further restriction $\alpha < n/\min\{p_1', p_2\}$ and the case (iii) when $\delta = n/\min\{p_1', p_2\}$. The conclusion is again that

$$a_k^B \leq ck^{-\frac{\alpha}{2n}\min\{p_1', p_2\}},$$

for some $c > 0$ independent of k.

In fact, it is possible to find s_0 such that $s_1 > s_0 > s_2$ and $\alpha < \delta' \equiv s_1 - s_0 - n(1/p_1 - 1/p_2) < n/\min\{p_1', p_2\}$, so that the embedding $B^{s_1}_{p_1q_1}(\alpha) \to B^{s_0}_{p_2q_2}$ falls under the case studied in 4.2.3. If we then use the estimate obtained there for the approximation numbers together with their multiplicativity, the composition

$$B^{s_1}_{p_1q_1}(\alpha) \to B^{s_0}_{p_2q_2} \to B^{s_2}_{p_2q_2}$$

leads us to the result announced above.

4.3 Some remarks

We have accomplished one of the goals stated in the introduction to this section, namely to get sharp estimates for the cases (ii), (iii) and (iv) considered there, with the exception of some critical situations. One just has to put together what has been obtained in 4.1 and 4.2 to get the picture (observe that the same estimates hold for a_k^F, as follows from standard arguments — cf. [7, 3.2]).

It is worth remarking that, as we have seen in 4.2.4, the situation $\delta = n/\min\{p_1', p_2\}$ in case (iii) is not a critical one and, in view of the results obtained, we can think of cases (iii) and (iv) as one case only:

$$0 < p_1 < 2 < p_2 < \infty \text{ and } \delta > \alpha \text{ and } \alpha \leq n/\min\{p_1', p_2\}.$$

This reformulation has the advantage that it makes it evident that the situation $\alpha = n/\min\{p_1', p_2\}$ is the only critical one here.

We want also to remark that, when comparing the result of this unified case with case (ii), namely

$$0 < p_1 < 2 < p_2 < \infty \text{ and } \alpha > \delta \text{ and } \delta \leq n/\min\{p_1', p_2\},$$

the roles of α and δ appear interchanged both in the definitions of the cases and in the estimates for the approximation numbers. Moreover, we can also unify (ii), (iii) and (iv) in those same two aspects, as we can state, in view of the results obtained, that, off the critical line $\alpha = \delta$,

$$\text{in the case } 0 < p_1 < 2 < p_2 < \infty \text{ and } \mu \equiv \min\{\alpha, \delta\} \leq n/\min\{p_1', p_2\},$$

if the last inequality is strict,

$$a_k^B \approx k^{-\frac{\mu}{2n}\min\{p_1', p_2\}}$$

(the same holds for a_k^F, of course).

It is interesting to note that the introduction of the parameter $\mu \equiv \min\{\alpha, \delta\}$ allows us also to unify the results in the cases for which Haroske [7] obtained sharp results, as can be observed in the following summary of the results known after our study (always assuming $\delta \neq \alpha$ and taking into consideration that we are forcing the entry in the second line below, because in the subcase of it given by $\mu = \alpha$ we only know that the exponent of the power is the correct one: as pointed out at the beginning of Section 4, we couldn't get rid of a perturbing factor of the type of a positive power of $\log(1+k)$ in the upper estimate):

$$a_k^B \approx a_k^F \approx \begin{cases} k^{-\frac{\mu}{n}} & \text{if } 0 < p_1 \leq p_2 \leq 2 \text{ or } 2 \leq p_1 \leq p_2 < \infty \\ k^{-\frac{\mu}{n}-\min\{\frac{1}{p_1}-\frac{1}{2},\frac{1}{2}-\frac{1}{p_2}\}} & \text{if } 0 < p_1 < 2 < p_2 < \infty \text{ and } \mu > \frac{n}{\min\{p_1',p_2\}} \\ k^{-\frac{\mu}{2n}\min\{p_1',p_2\}} & \text{if } 0 < p_1 < 2 < p_2 < \infty \text{ and } \mu < \frac{n}{\min\{p_1',p_2\}} \\ k^{-\frac{\mu}{n}+\frac{1}{p_2}-\frac{1}{p_1}} & \text{if } p_2 \leq p_1 \end{cases} .$$

If one compares this with the known behaviour for the approximation numbers of compact embeddings between spaces B_{pq}^s and F_{pq}^s on domains, one can't fail to notice that for $\delta < \alpha$ the estimates coincide. As a consequence we conclude that if $\delta < \alpha$ then the weight function has no influence in the estimates.

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