On fractals which are not so terrible [*]

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Abstract

The notion of NST domain and the closed related notion of ball condition, both topological in nature and quite useful within the theory of function spaces, are compared with each other (and with the older concept of porosity) and also with other notions of interest, like the concepts of d-set and of interior regular domain, which have a measure-theoretical nature. Also, after extending the idea of NST (not so terrible) to a larger class of sets, such property is studied in the context of anisotropic self-affine fractals.

1   Introduction

In [3, pp. 143-144] Frazier and Jawerth introduced the concept of NST (not so terrible) domain in order to characterize a property of certain functions belonging to Triebel-Lizorkin spaces. This follows a long trend, not to be discussed here, of imposing conditions on (or near the boundary of) domains in order to guarantee certain properties for function spaces on such domains. The interested reader may want to consult [7], [8] and [5], as well as the references given there, for the older material.

One of our points here is that the main idea underlying the concept of NST domain has been repeatedly introduced after the work [3] (and even before it,

under the name of porosity – cf. section 3 below), maybe in disguised form. So, for example, under mild hypotheses on a (fractal-to-be) set Γ, saying that Γ satisfies the so-called ball condition considered in [9, p. 142] is equivalent to saying that Γ^c is a NST domain. We shall detail this below, where the discussion will eventually lead us to define what should be meant by a not so terrible (NST) fractal.

Afterwards we give a direct proof that d-sets in $\mathbb{R}^n$ (see [5, pp. 28-32] or [9, p. 5]) are not so terrible fractals (or, what is the same, satisfy the ball condition) when $d < n$.

Next we study a subclass of the family of self-affine fractals and give sufficient conditions for an element of such subclass to be NST. Finally, we exhibit an example of a self-affine fractal which is really terrible (we mean, which is not NST), though it might not look like that at first glance.

2   Preliminary discussion

Definition 2.1 ([3, pp. 143-144]) *A domain (i.e., a non-empty open set) Ω of $\mathbb{R}^n$ is said to be NST (not so terrible), or to be in NST (thinking about NST as a denomination for the class of all NST domains in $\mathbb{R}^n$, for some fixed $n \in \mathbb{N}$), if there exists $\mu \in \mathbb{N}$ with the property that for any dyadic cube Q with side-length $\ell(Q) \leq 1$ and satisfying $\overline{Q} \cap \partial\Omega \neq \emptyset$, there exists a dyadic cube $P \subset Q$ with $\ell(P) = 2^{-\mu}\ell(Q)$ such that $\overset{\circ}{P} \cap \partial\Omega = \emptyset$.*

Of course, we will rule out, because of lack of interest, the case when $\Omega = \mathbb{R}^n$, that is, we shall only consider proper domains.

In the above definition, $\overset{\circ}{P}$, $\overline{Q}$ and $\partial\Omega$ stand, respectively, for the interior of P, the closure of Q and the boundary of Ω, as usual. A dyadic cube Q in $\mathbb{R}^n$ is a cube

$$Q_{\nu k} \equiv \{(x_1, \ldots, x_n) : 2^{-\nu}k_i \leq x_i < 2^{-\nu}(k_i + 1) \text{ for } i = 1, \ldots, n\}$$

for some $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. In particular, for each such ν, the family $\{Q_{\nu k} : k \in \mathbb{Z}^n\}$ constitutes a tessellation (without intersections) of $\mathbb{R}^n$.

In this paper, the word *cube*, even if not dyadic, is always used in the sense of *cube with sides parallel to the axes*.

The simplest examples of proper NST domains are the domains Ω where $\partial\Omega$ has just a finite set of points. On the other hand, if $\partial\Omega$ is "too crowded" then surely Ω is not NST. However, NST domains do not have a clear relationship with the cardinality of their boundaries: for example, as shall become clear along this paper, if $\partial\Omega$ is the Cantor set, then Ω is NST, though the Cantor set is uncountable; nevertheless, there are domains Ω not in NST for which $\partial\Omega$ is countable. We give an example in $\mathbb{R}$:

Let $\partial\Omega \equiv \{0\} \cup \{1/m : m \in \mathbb{N}\}$. Then $\Omega \notin NST$. In fact, assume that such Ω was in NST. Consider the sequence of dyadic cubes $Q_{\nu 1}$, $\nu \in \mathbb{N}$, which clearly intersect $\partial\Omega$, and a corresponding sequence of dyadic cubes $P(\nu) \subset Q_{\nu 1}$ with $\ell(P(\nu)) = 2^{-\mu}\ell(Q_{\nu 1})$ and $\overset{\circ}{P}(\nu) \cap \partial\Omega = \emptyset$. It is clear from the structure of $\partial\Omega$ that $\ell(P(\nu))$ must not exceed $1/2^{\nu-1}-1/(2^{\nu-1}+1)$, that is, $2^{-\nu+1}/(2^{\nu-1}+1)$, while $\ell(Q_{\nu 1}) = 2^{-\nu}$. Therefore $0 < 2^{-\mu} = \ell(P(\nu))/\ell(Q_{\nu 1}) \leq 2/(2^{\nu-1}+1) \to 0$ as ν goes to infinity, which is absurd!

3 NST and the ball condition

Definition 3.1 *A non-empty subset Γ of $\mathbb{R}^n$ such that $\overset{\circ}{\Gamma}= \emptyset$ is said to satisfy the ball condition if*

$$\exists_{\eta\in]0,1[}\forall_{x\in\Gamma}\forall_{r\in]0,1[}\exists_{y\in\mathbb{R}^n} B(y,\eta r) \subset B(x,r) \text{ and } B(y,\eta r) \cap \overline{\Gamma} = \emptyset,$$

where the notation $B(z,s)$ stands for the closed (Euclidean) ball centred at z with radius s.

This is a slight modification of the definition presented in [9, p. 142], where Γ was assumed to be Borel with zero Lebesgue measure. We decided to strip the definition off of any allusion to measure theory (and, since $|\Gamma| = 0 \Rightarrow \overset{\circ}{\Gamma}= \emptyset$, we are retaining all the sets considered to satisfy the ball condition by the definition given in [9]). Observe that the assumption $\overset{\circ}{\Gamma}= \emptyset$ is quite natural, as if $\overset{\circ}{\Gamma}\neq \emptyset$ then the proposition with quantifiers which follows in the definition cannot be fulfilled. Actually, it can even be proved that this proposition also implies that $|\overline{\Gamma}| = 0$. This can be seen as an immediate consequence of the fact that such a $\overline{\Gamma}$ has Hausdorff dimension strictly less

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than n, which in turn follows from the fact that $\overline{\Gamma}$ has strong porosity bounded away from 0 at all of its points, in the sense explained in [6, p. 156].

We would also like to mention that a set satisfying the ball condition is a particular kind of strongly porous set (cf. [6, p. 156]) and that, according to [6, p. 158], the notion of porosity goes back, at least in its weakest forms, to a work of Denjoy dated of 1920.

The following result can be proved by straightforward arguments:

Proposition 3.2 *Let Γ be a non-empty, closed subset of $\mathbb{R}^n$ with $\overset{\circ}{\Gamma}= \emptyset$. Then Γ satisfies the ball condition if, and only if, $\Gamma^c \in NST$.*

Remark 3.3 *Actually, the above statement is valid even without the assumption that Γ is closed, as long as we substitute the part of Γ^c being in NST by*

"there exists $\mu \in \mathbb{N}$ with the property that for any dyadic cube Q with side-length $\ell(Q) \leq 1$ and satisfying $\overline{Q} \cap \overline{\Gamma} \neq \emptyset$, there exists a dyadic cube $P \subset Q$ with $\ell(P) = 2^{-\mu}\ell(Q)$ such that $\overset{\circ}{P} \cap \overline{\Gamma} = \emptyset$",

where we can, as an alternative, write $\partial(\Gamma^c)$ or $\partial\Gamma$ in the places where $\overline{\Gamma}$ shows up.

The similarity between the quotation written above (with the possibility of replacing $\overline{\Gamma}$ by $\partial\Gamma$) and the definition of a NST domain prompts us for the following extension of Definition 2.1:

Definition 3.4 *Let Γ be a non-empty subset of $\mathbb{R}^n$ such that either $\overset{\circ}{\Gamma}= \Gamma$ or $\overset{\circ}{\Gamma}= \emptyset$. The set Γ is said to be NST (not so terrible), or to be in NST, if there exists $\mu \in \mathbb{N}$ with the property that for any dyadic cube Q with side-length $\ell(Q) \leq 1$ and satisfying $\overline{Q} \cap \partial\Gamma \neq \emptyset$, there exists a dyadic cube $P \subset Q$ with $\ell(P) = 2^{-\mu}\ell(Q)$ such that $\overset{\circ}{P} \cap \partial\Gamma = \emptyset$.*

Remark 3.5 (i) *Of course, in the case of $\overset{\circ}{\Gamma}= \emptyset$ we already know that $\partial\Gamma$ in the above definition can be replaced by $\partial(\Gamma^c)$ or $\overline{\Gamma}$. Also in that case, the remark above shows that $\Gamma \in NST$ if, and only if, Γ satisfies the ball condition.*

(ii) *With the above definition it is easy to see that a proper domain belongs to NST if, and only if, the same happens to its boundary; or, equivalently, if, and only if, its boundary satisfies the ball condition.*

We don't want to go into details, but we would also like to exhibit the relationship between the concept of NST set just introduced and the concept of open set condition as given in [10] (see this reference, for the definition): it is not difficult to see that a non-empty subset Γ of $\mathbb{R}^n$ with $\overset{\circ}{\Gamma} = \emptyset$ belongs to NST if, and only if, it satisfies such open set condition (apart from the imposition that Γ is Borel, made in [10]). Be aware, however, that the notion of open set condition used in [10] is different from the notion with the same name which we are going to consider later in this paper.

4 NST and d-sets

We recall the notion of d-set [5, pp. 28-32]:

Definition 4.1 *Let Γ be a non-empty closed subset of $\mathbb{R}^n$ and $d \in]0, n]$. Γ is said to be a d-set if*

$$\exists c_1, c_2 > 0 : \forall \gamma \in \Gamma, \forall 0 < r \leq 1, \ c_1 r^d \leq \mathcal{H}^d(B(\gamma, r) \cap \Gamma) \leq c_2 r^d, \quad (1)$$

where $\mathcal{H}^d$ stands for the d-dimensional Hausdorff measure on $\mathbb{R}^n$.

Remark 4.2 (i) *A d-set has Hausdorff dimension equal to d.*

(ii) *It is indifferent to use the Euclidean balls above or balls for the infinity norm (i.e., cubes), or to substitute $0 < r \leq 1$ by $0 < r \leq r_0$, for some fixed $r_0 > 0$, in the sense that these modifications do not change the class of sets to which the definition applies.*

This concept of d-set has been found very convenient in order to deal with function spaces on closed sets (see, for example, [5] and [9]). Be aware that the expression "d-set" has also been used with a different meaning in Fractal Geometry (cf., for example, [1]).

The following result can be derived from an assertion presented (without proof) in [4, Prop. 2]. We give a direct proof below.

Proposition 4.3 *Let Γ be a d-set, with $d < n$. Then $\Gamma \in NST$.*

Proof. From the properties of Hausdorff measure, it is clear that $\overset{\circ}{\Gamma} = \emptyset$.

Let Q_ν be a dyadic cube with side-length $2^{-\nu} \leq 1$ (that is, with $\nu \in \mathbb{N}_0$) and satisfying $\overline{Q_\nu} \cap \Gamma \neq \emptyset$. For each $\rho \in \mathbb{N}$ consider the tessellation of Q_ν by dyadic subcubes q with side-length $2^{-(\nu+\rho)}$. Clearly, there are $2^{\rho n}$ such subcubes.

Pick up only the subcubes q of that tessellation such that $\bar{q} \cap \Gamma \neq \emptyset$ and put them in some order: $q_1, \ldots, q_{n_\rho}$. In particular, n_ρ stands for the number of such smaller cubes. Next form a subsequence $(q_{\sigma(r)})_r$ from the preceding sequence, based on the following criteria (where cq stands for a cube with the same centre of q but with c times the side-length):

(i) q_1 is the first element of the subsequence;

(ii) if $q_{\sigma(r)}$ is an element of the subsequence and

$$J_r \equiv \{j \in \mathbb{N} : \forall s = 1, \ldots, r, 3\, \overset{\circ}{q}_{\sigma(s)} \cap q_{\sigma(r)+j} = \emptyset\} \neq \emptyset,$$

then $q_{\sigma(r+1)} \equiv q_{\sigma(r)+\min J_r}$;

(iii) the subsequence terminates when $J_r = \emptyset$.

Denote by m_ρ the number of cubes in this subsequence. It is clear that, for each cube q of the original sequence which remains in the subsequence, we may be throwing away as many cubes as the number of cubes of the tessellation which are neighbours of q (but no more than that). Since the number of such neighbours is at most $3^n - 1$, we can write

$$n_\rho \leq m_\rho + m_\rho(3^n - 1) = 3^n m_\rho. \tag{2}$$

Recall that each $\overline{q_{\sigma(r)}}$ contains an element of Γ. Fix one such member of Γ (call it x_r) in each one of the cubes $\overline{q_{\sigma(r)}}$ and denote by K_r the closed cubes centred at those x_r and with side-length $2^{-(\nu+\rho+1)}$. Observe that $K_r \subset 2\, \overset{\circ}{q}_{\sigma(r)}$ and that the m_ρ cubes $2\, \overset{\circ}{q}_{\sigma(r)}$ are pairwise disjoint.

Fix also a point in $\overline{Q_\nu} \cap \Gamma$ and consider the closed cube K centred at that point and with side-length $3 \times 2^{-\nu}$, so that, in particular, $\bigcup_{r=1}^{m_\rho} 2\, \overset{\circ}{q}_{\sigma(r)} \subset K$.

Now we use the hypothesis that Γ is a d-set in order to write

$$c_2(3 \times 2^{-\nu-1})^d \;\geq\; \mathcal{H}^d(K \cap \Gamma) \geq \mathcal{H}^d\left(\bigcup_{r=1}^{m_\rho} 2 \, \overset{\circ}{q_{\sigma(r)}} \cap \Gamma\right)$$

$$= \sum_{r=1}^{m_\rho} \mathcal{H}^d(2 \, \overset{\circ}{q_{\sigma(r)}} \cap \Gamma) \geq \sum_{r=1}^{m_\rho} \mathcal{H}^d(K_r \cap \Gamma)$$

$$\geq \sum_{r=1}^{m_\rho} c_1 2^{-(\nu+\rho+2)d} = m_\rho c_1 2^{-\nu d} 2^{-(\rho+2)d},$$

from which follows, taking care of (2), that

$$n_\rho \leq c_3 2^{\rho d}, \tag{3}$$

where $c_3 > 0$ depends only on n, d and Γ.

Since there are exactly $2^{\rho n}$ dyadic cubes of side-length $2^{-(\nu+\rho)}$ in Q_ν, then the number of such cubes q such that $\overline{q} \cap \Gamma = \emptyset$ is $2^{\rho n} - n_\rho$. Using (3) we can then guarantee that the number of such cubes is bounded below by $2^{\rho n} - c_3 2^{\rho d}$. As this goes to infinity with ρ (here we have used the hypothesis $d < n$), we can be sure that there is a value ρ_0 of ρ (depending only on n, d and Γ) such that there is at least one dyadic subcube q of side-length $2^{-\rho_0} \ell(Q_\nu)$ in Q_ν such that $\overset{\circ}{q} \cap \Gamma = \emptyset$.

This completes the proof that $\Gamma \in NST$ (with $\mu = \rho_0$ in Definition 3.4). $\square$

Remark 4.4 *Even if one considers only not so terrible sets Γ which are closed, there are some which are not d-sets, for any d. Take, for example, $\Gamma = \{0\} \cup \{2^{-m} : m \in \mathbb{N}_0\}$ in $\mathbb{R}$. Of course, no open NST set can be a d-set with $d < n$ (even if we omit the assumption of being closed in the definition of d-set, as we will do whenever necessary). Observe also that there are open NST sets which are not open n-sets (consider, for example, a domain with an outward cusp, as in Fig. 5 of [11, p. 657]). On the positive side, it's not difficult to see that, given a NST domain Ω, the domain $(\partial\Omega)^c$ is an open n-set.*

Clearly, a n-set might not be a NST set in the sense of Definition 3.4: a simple example is given by a closed cube in $\mathbb{R}^3$, just because this cube is neither an open set nor has empty interior. However, the difference between the two concepts is deeper than this: if one tries to apply (1) to open instead of closed sets, one might find open sets which are n-sets in this modified sense but which do not belong to NST: a simple example is given by the open Γ in $\mathbb{R}$ such that $\Gamma^c = \partial\Gamma = \{0\} \cup \{1/m : m \in \mathbb{N}\}$ (recall that we have already proved that such Γ is not NST – see section 2). We give a more complicated example in two dimensions later on. As for n-sets (closed or not) Γ such that $\overset{\circ}{\Gamma} = \emptyset$, they cannot be in NST: on one hand, being n-sets, they must satisfy $|\Gamma| > 0$; on the other hand, if this is true then Γ cannot satisfy the ball condition (see comments after Definition 3.1); since $\overset{\circ}{\Gamma} = \emptyset$, it follows then by Remark 3.5(i) that $\Gamma \notin NST$.

We should also like to mention that a domain Ω in $\mathbb{R}^n$ is an open n-set if, and only if, it is an interior regular domain in the sense of [11], apart from the imposition of boundedness of Ω and of the equality $\Omega = \overset{\circ}{\overline{\Omega}}$, made there. More explicitly, if, and only if, there exists a positive c with the property that for any $x \in \partial\Omega$ and any cube Q centred at x with side-length at most 1, $|\Omega \cap Q| \geq c|Q|$.

5 NST and self-affine fractals

Let $N \geq 2$ be a natural number and let $A_1, \ldots, A_N$ be affine contractions on $\mathbb{R}^n$. That is, for each $l = 1, \ldots, N$, A_l is an affine map (i.e.,

$$A_l x = T_l x + b_l, \ \forall x \in \mathbb{R}^n, \tag{4}$$

where $b_l \in \mathbb{R}^n$ and T_l is a linear transformation on $\mathbb{R}^n$) and A_l is a contraction (i.e., $\exists\, 0 < c_l < 1$:

$$|A_l x - A_l y| \leq c_l |x - y|, \ \forall x, y \in \mathbb{R}^n).$$

Of course, in view of (4), the last property can equivalently be described by the existence of $0 < c_l < 1$ such that $|T_l x| \leq c_l |x|, \forall x \in \mathbb{R}^n$, or just by stating that $\|T_l\| < 1$.

Definition 5.1 *Given N affine contractions $A_1, \ldots, A_N$, the unique non-empty compact subset Γ of $\mathbb{R}^n$ satisfying the equality*

$$\Gamma = \bigcup_{l=1}^{N} A_l \Gamma \tag{5}$$

is called the self-affine fractal generated by those N maps.

Remark 5.2 (i) *The word "fractal" is being used loosely here – we could as well have written "set" instead.*

(ii) *This definition includes a proposition, namely that there is one and only one non-empty compact set Γ satisfying (5). This is indeed the case: see [2, p. 114], where it is also shown that Γ can be obtained as*

$$\Gamma = \bigcap_{k=1}^{\infty} A^k E, \tag{6}$$

for any non-empty compact subset E of $\mathbb{R}^n$ such that $A_l E \subset E, \forall l = 1, \ldots, N$. In (6) $A^k E$ is defined in the following way:

$$A^0 E = E ; \quad AE = \cup_{l=1}^{N} A_l E ; \quad A^k E = A(A^{k-1} E), \ \forall k \in \mathbb{N}. \tag{7}$$

(iii) *For examples and pictures, the interested reader is referred to [2].*

The case when the affine contractions $A_1, \ldots, A_N$ are similarities, that is, satisfy

$$|A_l x - A_l y| = c_l |x - y|, \ \forall x, y \in \mathbb{R}^n, \tag{8}$$

for some $0 < c_l < 1, \forall l = 1, \ldots, N$, is well studied, mainly if those affine contractions are also supposed to satisfy the open set condition, which means that there exists a non-empty open set $\mathcal{U}$ in $\mathbb{R}^n$ such that

$$A\mathcal{U} \subset \mathcal{U} \quad \text{and} \quad A_l \mathcal{U} \cap A_h \mathcal{U} = \emptyset, \quad \text{for} \ \ l \neq h \tag{9}$$

(with A as defined in (7)). Thus, the following is known (see [6, p. 67]):

Proposition 5.3 *Let $A_1, \ldots, A_N$ be similarities satisfying the open set condition. Then the self-affine fractal generated by these contractions is a d-set, where d is the unique positive number satisfying*

$$\sum_{l=1}^{N} c_l^d = 1$$

(c_l as in (8)).

In the case when $d < n$ we can immediately conclude, on the basis of Proposition 4.3, that a self-affine fractal such as one in the preceding proposition is NST.

Our next aim is to consider a class of self-affine fractals of sufficient interest (and where the contractions are not necessarily similarities) and establish easily verifiable conditions which guarantee that a particular element of that class is NST. Since we shall also show that not all elements of such class are NST, what follows can also be considered as a contribution to understand how far from the class NST is the class of self-affine fractals.

Let Q be the following closed unit cube in $\mathbb{R}^n$:

$$Q \equiv \{x \equiv (x_j)_{j=1}^{n} : 0 \le x_j \le 1, \forall j = 1, \ldots, n\}.$$

Our class of self-affine fractals should contain only those which are generated by surjective ($\Leftrightarrow$ injective in this context) affine contractions $A_l, l = 1, \ldots, N$, such that

$$A_l Q \subset Q, \ l = 1, \ldots, N, \tag{10}$$

$$A_l \overset{\circ}{Q} \cap A_m \overset{\circ}{Q} = \emptyset \ \text{ if } \ l \ne m \, ; \ l, m = 1, \ldots, N, \tag{11}$$

and

$$\sum_{l=1}^{N} |A_l Q| < 1, \tag{12}$$

where here $|\cdot|$ stands for Lebesgue measure in $\mathbb{R}^n$.

According to our Remark 5.2(ii) and (10), the self-affine fractal generated by a particular family of A_l as above is the set Γ given by

$$\Gamma = \bigcap_{k=1}^{\infty} A^k Q,$$

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where A^k has the same meaning as in (7). In particular,

$$A^k Q = \bigcup_{1 \le l_1,\ldots,l_k \le N} A_{l_1} \ldots A_{l_k} Q \ , \quad k \in \mathbb{N}.$$

Note that the hypotheses given for A_l also imply that such a Γ satisfies the open set condition (9) with $\mathcal{U} = \overset{\circ}{Q}$.

Lemma 5.4 *Each self-affine fractal of the class now introduced has zero Lebesgue measure.*

Proof.

$$
\begin{aligned}
0 \le |\Gamma| \ &\le \ |A^k Q| \le \sum_{1 \le l_1,\ldots,l_k \le N} |A_{l_1} \ldots A_{l_k} Q| \\
&= \sum_{1 \le l_1,\ldots,l_k \le N} |\det T_{l_1}| \ldots |\det T_{l_k}| \\
&= \left(\sum_{1 \le l_1 \le N} |\det T_{l_1}| \right) \ldots \left(\sum_{1 \le l_k \le N} |\det T_{l_k}| \right) \\
&= \ r^k
\end{aligned}
$$

where $r \equiv \sum_{l=1}^{N} |A_l Q|$, which, by (12), we know it is less than 1. The conclusion follows immediately by letting k go to ∞. $\square$

Remark 5.5 *As a consequence, such sets Γ have empty interior. Therefore, a particular set Γ of that class is NST if, and only if, satisfies the ball condition – cf. Remark 3.5(i).*

The preceding class is still too large for what we intend to do. We restrict further attention to the so-called anisotropic fractals, in the sense of [9, p. 14] (though here n can be any natural number), which is a subclass of the one we have considered above. For anisotropic fractals, the affine contractions A_l have the form

$$A_l : (x_j)_{j=1}^n \longmapsto (\eta_j^{(l)} r_j^{(l)} x_j)_{j=1}^n + (b_j^{(l)})_{j=1}^n, \tag{13}$$

where $\eta_j^{(l)} \in \{-1, 1\}$, $r_j^{(l)} \in {]0, 1[}$ and $b_j^{(l)}$, $j = 1, \ldots, n$ are given constants for each $l = 1, \ldots, N$. We are, in fact, going to consider only such type of fractals in what follows.

Theorem 5.6 *Let Γ be an anisotropic self-affine fractal. Let $A_l, l = 1, 2, \ldots, N$, denote the affine contractions (13) which generate Γ. Assume that for every $i \in \{1, 2, \ldots, n\}$, every $l \in \{1, 2, \ldots, N\}$ and every $x \equiv (x_j)_{j=1}^n \in A_l Q$, there exist $y \equiv (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$ and an open cube $q \subset Q$ such that $y \in \bar{q}$ and*

$$q \cap \left(\bigcup_{\ell=1}^N A_\ell Q \right) = \emptyset.$$

Then Γ is NST.

Proof. *Step 1.* Observe that there is a $\mu > 0$ such that the side-length of any q as above can be taken to be $\geq \mu$. Also, given that q is open, $q \subset Q$ is equivalent to $q \subset \overset{\circ}{Q}$.

According to Remark 5.5, we must prove that Γ satisfies the ball condition.

Consider any $x \equiv (x_j)_{j=1}^n \in \Gamma = \bigcap_{k=1}^\infty A^k Q = \bigcap_{k=1}^\infty \bigcup_{1 \leq l_1, \ldots, l_k \leq N} A_{l_1} \ldots A_{l_k} Q$ and any closed ball $B(x, r)$ centred at x and with radius $r \in]0, 1[$. Note that $x \in A^k Q, \forall k \in \mathbb{N}$, and that, for each k considered, x belongs to some $A_{l_1} \ldots A_{l_k} Q$, where $1 \leq l_1, \ldots, l_k \leq N$.

Step 2. Let $\alpha \equiv \min\{r_j^{(l)} : j \in \{1, \ldots, n\}, l \in \{1, \ldots, N\}\}$, $\beta \equiv \max\{r_j^{(l)} : j \in \{1, \ldots, n\}, l \in \{1, \ldots, N\}\}$ and $\theta \equiv (\alpha\beta)/(\sqrt{n}\alpha + \beta)$. Note that $\theta < \alpha$, whence $r_j^{(l)} > \theta r, \forall j \in \{1, \ldots, n\}, \forall l \in \{1, \ldots, N\}$. On the other hand, the sequence $(r_j^{(l_1)} \ldots r_j^{(l_m)})_{m \in \mathbb{N}}$, viewed as a sequence of functions of $(l_s)_{s \in \mathbb{N}}$ and j, tends to zero uniformly on the set $\{((l_s)_{s \in \mathbb{N}}, j) : (l_s)_{s \in \mathbb{N}} \subset \{1, \ldots, N\}, j \in \{1, \ldots, n\}\}$, which follows from inequalities $0 < r_j^{(l_1)} \ldots r_j^{(l_m)} \leq \beta^m$ and the fact that $\beta < 1$. Therefore it is possible to find $k \in \mathbb{N}$ such that, for any such j and $(l_s)_{s \in \mathbb{N}}$, it holds $r_j^{(l_1)} \ldots r_j^{(l_k)} \leq \theta r$.

Fix now $l_1, \ldots, l_k$ in such a way that for the given x we have

$$x \in A_{l_1} \ldots A_{l_k} Q. \tag{14}$$

Since, for each j, $r_j^{(l_1)} > \theta r$ and $r_j^{(l_1)} \ldots r_j^{(l_k)} \leq \theta r$, there is a first natural $k_j \in [2, k]$ such that the product $r_j^{(l_1)} \ldots r_j^{(l_{k_j})}$ becomes $\leq \theta r$. Let $k_0 \equiv \min_{1 \leq j \leq n} k_j$, so that $r_j^{(l_1)} \ldots r_j^{(l_{k_0-1})} > \theta r$ for all $j \in \{1, \ldots, n\}$. Among all j's such that $r_j^{(l_1)} \ldots r_j^{(l_{k_0})} \leq \theta r$ choose one – call it i – such that the product is minimum.

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For convenience of writing, we shall denote k_0 again by k. So, in what follows the $k \in \mathbb{N}$ and the $i \in \{1, \ldots, n\}$ were chosen such that

$$r_i^{(l_1)} \ldots r_i^{(l_k)} \le \theta r \tag{15}$$

and

$$\forall j \in \{1, \ldots, n\}, \ r_j^{(l_1)} \ldots r_j^{(l_{k-1})} > \theta r, \ r_j^{(l_1)} \ldots r_j^{(l_k)} \ge r_i^{(l_1)} \ldots r_i^{(l_k)}. \tag{16}$$

Step 3. Note that (14) still holds for this new k (this is an easy consequence of (10)), so that there exists $x' \equiv (x'_j)_{j=1}^n \in A_{l_k} Q$ such that $x = A_{l_1} \ldots A_{l_{k-1}} x'$. From the hypothesis, there exists $y' = (x'_1, \ldots, x'_{i-1}, y'_i, x'_{i+1}, \ldots, x'_n)$ and an open cube $q' \subset \overset{\circ}{Q}$ such that $y' \in \overline{q'}$ and

$$q' \cap \left(\bigcup_{l=1}^N A_l Q \right) = \emptyset, \tag{17}$$

from which follows that

$$A_{l_1} \ldots A_{l_{k-1}} q' \subset A_{l_1} \ldots A_{l_{k-1}} \overset{\circ}{Q} \tag{18}$$

and

$$(A_{l_1} \ldots A_{l_{k-1}} q') \cap \left(\bigcup_{l=1}^N A_{l_1} \ldots A_{l_{k-1}} A_l Q \right) = \emptyset. \tag{19}$$

As a consequence,

$$(A_{l_1} \ldots A_{l_{k-1}} q') \cap \Gamma = \emptyset. \tag{20}$$

In fact, if $z \in A_{l_1} \ldots A_{l_{k-1}} q'$ then, by (18), $z \in A_{l_1} \ldots A_{l_{k-1}} \overset{\circ}{Q}$, and therefore $z \notin A_{h_1} \ldots A_{h_{k-1}} \overset{\circ}{Q}$, for any $(h_1, \ldots, h_{k-1}) \ne (l_1, \ldots, l_{k-1})$ – this is an easy consequence of (11).

Step 4. Consider $y \equiv A_{l_1} \ldots A_{l_{k-1}} y'$. Note that $q' = y' + q''$, where $q'' \equiv -y' + q'$ is also an open cube. Define $q''' \equiv \{r_i^{(l_1)} \ldots r_i^{(l_{k-1})} \alpha \beta^{-1} (\eta_j^{(l_1)} \ldots \eta_j^{(l_{k-1})} z_j)_{j=1}^n : (z_j)_{j=1}^n \in q''\}$, which is also an open cube, now with side-length equal to $r_i^{(l_1)} \ldots r_i^{(l_{k-1})} \alpha \beta^{-1}$ times the side-length of q'. The same happens to the open cube $q \equiv y + q'''$. We claim that

$$y \in \overline{q} \tag{21}$$

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and that

$$q \subset A_{l_1} \dots A_{l_{k-1}} q'. \tag{22}$$

Assertion (21) is clear, as $y' \in \overline{q'} \Rightarrow 0 \in \overline{-y' + q'} \Rightarrow 0 \in \overline{q'''} \Rightarrow y \in \overline{y + q'''}$. As to (22), observe that, given $z \in q$, $z = y + z'''$ for some $z''' \in q'''$, $z = A_{l_1} \dots A_{l_{k-1}} y' + r_i^{(l_1)} \dots r_i^{(l_{k-1})} \alpha \beta^{-1} (\eta_j^{(l_1)} \dots \eta_j^{(l_{k-1})} z_j'')_{j=1}^n$ for some $(z_j'')_{j=1}^n \in q''$, $z = (\eta_j^{(l_1)} \dots \eta_j^{(l_{k-1})} r_j^{(l_1)} \dots r_j^{(l_{k-1})} y_j')_{j=1}^n + c_{l_1,\dots,l_{k-1}} + (\eta_j^{(l_1)} \dots \eta_j^{(l_{k-1})} r_j^{(l_1)} \dots r_j^{(l_{k-1})} \lambda_j z_j'')_{j=1}^n$, where $\lambda_j \equiv r_i^{(l_1)} \dots r_i^{(l_{k-1})} \alpha \beta^{-1} / (r_j^{(l_1)} \dots r_j^{(l_{k-1})})$ and $c_{l_1,\dots,l_{k-1}}$ is the independent vector of the affine map $A_{l_1} \dots A_{l_{k-1}}$ (as the b_l in (4)). That is, any $z \in q$ can be written as $z = (\eta_j^{(l_1)} \dots \eta_j^{(l_{k-1})} r_j^{(l_1)} \dots r_j^{(l_{k-1})} (y_j' + \lambda_j z_j''))_{j=1}^n + c_{l_1,\dots,l_{k-1}}$ $= A_{l_1} \dots A_{l_{k-1}} (y' + (\lambda_j z_j'')_{j=1}^n)$, for some $z'' \equiv (z_j'')_{j=1}^n \in q''$. Since $0 < \lambda_j \le 1$, j=1, ..., n (cf. (16) and the definitions of α and β, given at the beginning of Step 2), $z'' \in q''$ and $0 \in \overline{q''}$, then $(\lambda_j z_j'')_{j=1}^n$ is also in q'' (recall that we are only considering cubes with sides parallel to the axes) and (22) now follows easily.

As a consequence of (20) and (22), we have that

$$q \cap \Gamma = \emptyset. \tag{23}$$

Step 5. Observe now that, given any $z \in \overline{q}$,

$$
\begin{aligned}
|z - x| &\le |z - y| + |y - x| \le \sqrt{n}\, r_i^{(l_1)} \dots r_i^{(l_{k-1})} \alpha \beta^{-1} + r_i^{(l_1)} \dots r_i^{(l_{k-1})} |y_i' - x_i'| \\
&\le r_i^{(l_1)} \dots r_i^{(l_{k-1})} (\sqrt{n} \alpha \beta^{-1} + 1) \le r_i^{(l_1)} \dots r_i^{(l_{k-1})} r_i^{(l_k)} \alpha^{-1} (\sqrt{n} \alpha \beta^{-1} + 1) \\
&\le \theta r \alpha^{-1} (\sqrt{n} \alpha \beta^{-1} + 1) = r
\end{aligned}
$$

(we have used (15) in the last inequality), so that

$$\overline{q} \subset B(x, r).$$

Finally, let c stand for the centre of q and consider the ball $B(c, \ell(q)/2)$. We have that this ball is in $\overline{q}$, therefore also in $B(x, r)$, and has radius $r_i^{(l_1)} \dots r_i^{(l_{k-1})} \alpha \beta^{-1} \ell(q')/2$, which, by (16) and the first phrase in Step 1, is strictly bounded below by $\theta r \alpha \beta^{-1} \mu / 2$. Hence

$$B(c, \theta \alpha \beta^{-1} \mu r / 2) \subset q \subset B(x, r)$$

and, due to (23),

$$B(c, \theta \alpha \beta^{-1} \mu r / 2) \cap \Gamma = \emptyset.$$

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This accomplishes the proof that Γ satisfies the ball condition, with $\eta = \theta\alpha\beta^{-1}\mu/2$ in Definition 3.1. $\square$

Corollary 5.7 *Let Γ be an anisotropic self-affine fractal. Let $A_l, l = 1, \ldots, N$, denote the affine contractions (13) which generate Γ. Assume that there is $k_0 \in \mathbb{N}$ such that for every $i \in \{1, \ldots, n\}$, every $l_1, \ldots, l_{k_0} \in \{1, \ldots, N\}$ and every $x \equiv (x_j)_{j=1}^n \in A_{l_1} \ldots A_{l_{k_0}} Q$, there exist $y \equiv (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$ and an open cube $q \subset Q$ such that $y \in \bar{q}$ and*

$$q \cap A^{k_0} Q = \emptyset.$$

Then Γ is NST.

Proof. Consider the self-affine fractal Γ' generated by the family $\{A_{l_1 \ldots l_{k_0}} : l_1, \ldots, l_{k_0} \in \{1, \ldots, N\}\}$ of affine contractions $A_{l_1 \ldots l_{k_0}} \equiv A_{l_1} \ldots A_{l_{k_0}}$.

It's not difficult to see that Γ' is an anisotropic fractal fitting the conditions imposed in Theorem 5.6, so that we can conclude that Γ' is NST.

Now we just have to note that $\Gamma = \Gamma'$. $\square$

In the following examples of anisotropic self-affine fractals in $\mathbb{R}^2$, the affine contractions A_l are represented by means of the images $A_l Q$, which, in turn, are represented by the shaded rectangles in the corresponding figure.
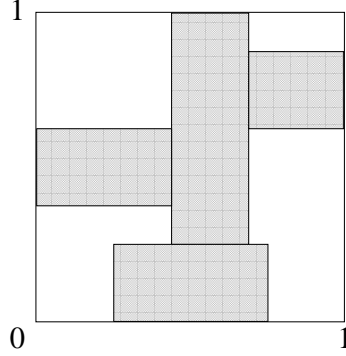
Example 1: The anisotropic fractal generated by the affine contractions represented in Figure 1 is NST.

Figure 1:

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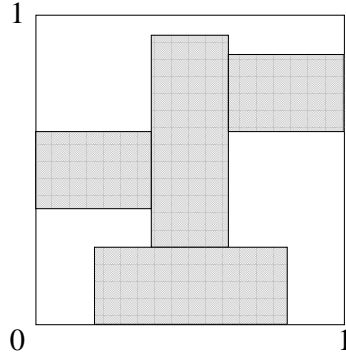
Example 2: An anisotropic fractal to which the corollary (with $k_0 = 3$) – but not the theorem – can be applied is generated by the affine contractions represented in Figure 2.

Figure 2:

We can weaken the hypotheses of Theorem 5.6 if one is dealing with a special subclass of the class of anisotropic fractals. We start with some preparations:

With the same notations as in (13), given A_l, $l = 1, \ldots, N$, if $i \in \{1, \ldots, n\}$ is such that $r_i^{(l)} = \min_{1 \leq j \leq n} r_j^{(l)}$, we say that the i-axis is an axis of stronger contraction (for the affine contraction A_l).

Definition 5.8 *An anisotropic fractal is called a CASC-fractal if the affine contractions of a generating family $\{A_l : l \in \{1, \ldots, N\}\}$ have a Common Axis of Stronger Contraction, that is, if $\exists i \in \{1, \ldots, n\} : \forall l = 1, \ldots, N$, $\min_{1 \leq j \leq n} r_j^{(l)} = r_i^{(l)}$.*

Theorem 5.9 *Let Γ be a CASC-fractal. Let $A_l, l = 1, \ldots, N$, denote the affine contractions (13) which generate Γ and let the i-axis be a common axis of stronger contraction. Assume that, for every $l \in \{1, \ldots, N\}$ and every $x \equiv (x_j)_{j=1}^n \in A_l Q$, there exist $y \equiv (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$ and an open cube $q \subset Q$ such that $y \in \bar{q}$ and*

$$q \cap \left(\bigcup_{\ell=1}^{N} A_\ell Q \right) = \emptyset.$$

Then Γ is NST.

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Proof. Since the proof is very similar to the proof of Theorem 5.6, we are only going to point out the necessary modifications to the latter.

Step 1 is the same.

Step 2: To begin with, θ is now defined as $\alpha/(\sqrt{n}+1)$. The parameter α itself can be defined more simply as $\min\{r_i^{(l)} : l \in \{1,\ldots,N\}\}$, because of the role played by the number i, which is here given from the very beginning. As to β, it is enough now to define it by $\max\{r_i^{(l)} : l \in \{1,\ldots,N\}\}$. Since i is given *a priori*, in this step we can fix $j = i$, so that there is no point in considering k_0 – or, if one prefers, just take $k_0 = k_i$ – and one should suppress the part where the i is chosen. We again use the letter k to denote k_0 (that is, k_i), and the last part of Step 2 should now read... in what follows the $k \in \mathbb{N}$ was chosen such that

$$r_i^{(l_1)} \ldots r_i^{(l_k)} \le \theta r \tag{24}$$

and

$$r_i^{(l_1)} \ldots r_i^{(l_{k-1})} > \theta r. \tag{25}$$

Step 3 is the same.

Step 4: Suppress the factor $\alpha\beta^{-1}$ in all considerations. In particular, λ_j now reads $r_i^{(l_1)} \ldots r_i^{(l_{k-1})}/(r_j^{(l_1)} \ldots r_j^{(l_{k-1})})$. The justification that $0 < \lambda_j \le 1$ must be accordingly modified (but it's trivial).

Step 5: Suppress the factor $\alpha\beta^{-1}$ in all considerations. $\square$

Corollary 5.10 *Let* Γ *be a CASC-fractal. Let* $A_l, l = 1,\ldots,N,$ *denote the affine contractions (13) which generate* Γ *and let the* i*-axis be a common axis of stronger contraction. Assume that there is* $k_0 \in \mathbb{N}$ *such that for every* $l_1,\ldots,l_{k_0} \in \{1,\ldots,N\}$ *and every* $x \equiv (x_j)_{j=1}^n \in A_{l_1}\ldots A_{l_{k_0}}Q,$ *there exist* $y \equiv (x_1,\ldots,x_{i-1}, y_i, x_{i+1},\ldots,x_n)$ *and an open cube* $q \subset Q$ *such that* $y \in \bar{q}$ *and*

$$q \cap A^{k_0}Q = \emptyset.$$

Then Γ *is NST.*

The proof follows as easily as the proof of Corollary 5.7

In the following examples of CASC-fractals in $\mathbb{R}^2$, the affine contractions A_l are represented by means of the images $A_l Q$, which, in turn, are represented by the shaded rectangles in the corresponding figure.
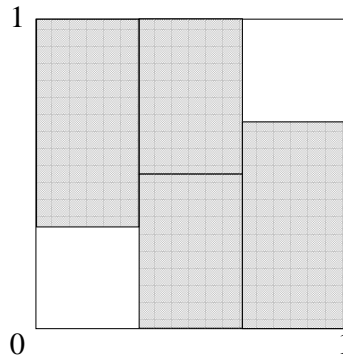
Example 3: A CASC-fractal to which Theorem 5.9 – but not Theorem 5.6 nor Corollary 5.7 – can be applied is generated by the affine contractions represented in Figure 3.

Figure 3:
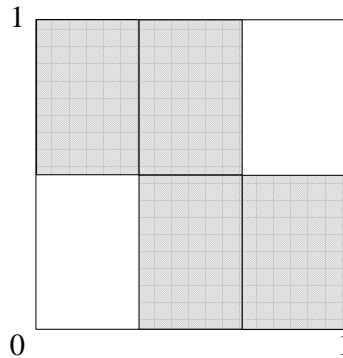
Example 4: A CASC-fractal to which Corollary 5.10 – but not Theorems 5.6 and 5.9 nor Corollary 5.7 – can be applied is generated by the affine contractions represented in Figure 4.
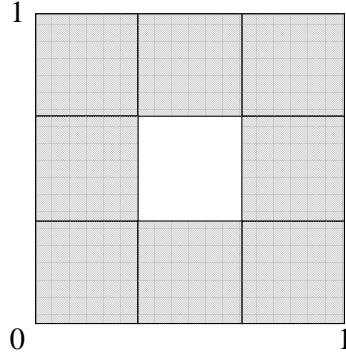
Figure 4:

Of course, Theorems 5.6, 5.9 and Corollaries 5.7, 5.10 only give sufficient conditions in order for an anisotropic fractal to be NST:

Example 5: The 2-dimensional Cantor set generated by the similarities represented in Figure 5 is a CASC-fractal to which none of those results apply; nevertheless, it is NST (cf. Proposition 5.3 and comments afterwards).
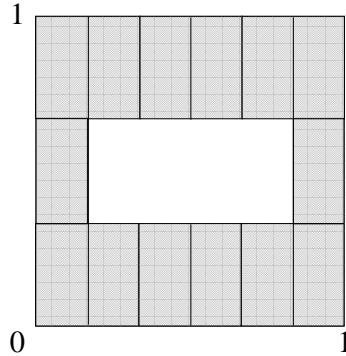
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Figure 5:

Some conditions must, however, be imposed in order for an anisotropic (even if CASC-) fractal to be NST, as the following example shows.

Example 6: Consider the anisotropic 2-dimensional Cantor-like set Γ generated by the affine contractions represented in Figure 6.

Figure 6:

Clearly, this is even a CASC-fractal. We shall see, however, that it is not NST:

First observe that, denoting by R the free space open rectangle in the middle of Q in Figure 6 (that is, $R = \overset{\circ}{Q} \setminus \bigcup_{l=1}^{14} A_l Q$),

$$\overset{\circ}{Q} \setminus \Gamma = R \cup \bigcup_{k=1}^{\infty} \bigcup_{1 \leq l_1,\dots,l_k \leq 14} A_{l_1} \dots A_{l_k} R.$$

Calling $A_{l_1} \dots A_{l_k} R$ free space rectangles of the k-th generation and using the usual convention to denote coordinates in dimension 2, note that, below the

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line $y = 3^{-(k+1)}$ in Figure 6 there can be no free space rectangles of generation k popping in. On the other hand, the basis of each free space rectangle of generation k has length $(2/3)6^{-k}$, so that the higher the generation, the smaller the length of the bases of its free space rectangles.

Consider then the sequence $((1/3, 3^{-(k+1)}))_{k \in \mathbb{N}}$ of points of Γ and a corresponding sequence of closed balls B_k of radius $3^{-(k+1)}$ centred at those points. Observe that each such ball is contained in Q and stays below the line $y = 3^{-k}$. Therefore, the only points of that ball which are not in Γ must be in free space rectangles of k-th or higher generation. Since these are pairwise disjoint, it is not possible to find a closed ball contained in $B_k \setminus \Gamma$ and with radius greater than $(1/3)6^{-k}$. And since the ratio $((1/3)6^{-k})/(3^{-(k+1)})$, that is, 2^{-k}, tends to 0 as k goes to infinity, it is not possible to find $\eta \in]0, 1[$ such that, for any $k \in \mathbb{N}$ and for some $y_k \in \mathbb{R}^2$, $B(y_k, \eta 3^{-(k+1)}) \subset B_k$ and $B(y_k, \eta 3^{-(k+1)}) \cap \overline{\Gamma} = \emptyset$.

This shows that such a Γ does not satisfy the ball condition. That Γ is not NST follows now from Remark 5.5.

Remark 5.11 (i) *This also serves as an example of a fractal satisfying the open set condition (see (9)) but which is not in NST, so that in this way we complement what has been said after Remark 3.5.*

(ii) *The Γ of Example 6 is not a d-set. This is clear for $d < n$, due to Proposition 4.3; for $d = n$ it is also clear from the fact that $|\Gamma| = 0$ (Lemma 5.4).*

(iii) *Consider the open set Γ^c, with Γ the fractal in Example 6. From this and Proposition 3.2 it follows that $\Gamma^c \notin NST$. Since $|\Gamma| = 0$, Γ^c is an open n-set and therefore we have found another example (not as trivial as the one exhibited after Remark 4.4) of an open n-set which is not NST.*

Example 7: From Theorem 5.9 it is clear that the modification of Figure 6 given in Figure 7 yields, in the usual way, a representation of a CASC-fractal which is NST.

Figure 7:
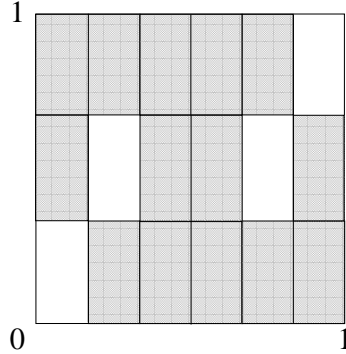
Remark 5.12 *This together with Example 6 show that, for anisotropic fractals, the property of being NST or not has nothing to do with the so-called affine dimension of fractals, as defined in [9]. Those two in Examples 6 and 7 have the same affine dimension (just because the number of shaded rectangles of the same size is the same, irrespective of their positions) – namely $(2+2\log_2 7)/(1+2\log_2 3)$ – while one is NST and the other is not (the results proved above show that the relative position of the shaded rectangles can make a difference).*

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