Eigenvalue asymptotics of the Stokes operator for fractal domains *

António M. Caetano

0 Introduction

The main aim of this paper is to prove for the eigenvalues of the Stokes operator certain results corresponding to some that are well-known for the eigenvalues of the Dirichlet Laplacian (and of other elliptic operators) — cf. [8].

We take *a priori* for the underlying set Ω any non-empty bounded open subset of $\mathbb{R}^n$, no matter how irregular its boundary $\partial\Omega$ is. In the case of the Dirichlet Laplacian, it is known that the fractality of $\partial\Omega$ plays an important role in the asymptotics of the eigenvalues of the operator. Here we want to show that the same seems to be true for the asymptotics of the eigenvalues of the Stokes operator.

As far as we know, the results that have been obtained for these asymptotics deal only with the situation when $\partial\Omega$ is smooth. Thus we have the determination by Métivier [11] of the first term for the counting function $N(\lambda)$ associated with the problem in the case Ω is Lipschitz, namely

$$N(\lambda) \sim \frac{|\Omega|_n}{(2\pi)^n}(n-1)|B^n|_n\lambda^{n/2} \qquad as \quad \lambda \to \infty, \tag{0.1}$$

where $N(\lambda)$ is defined as the number of eigenvalues not exceeding λ, $|\cdot|_n$ stands

*1991 *Mathematics Subject Classification* 35P20 for Lebesgue measure in $\mathbb{R}^n$ and B^n is used to denote the Euclidean unit ball of $\mathbb{R}^n$.

Formula (0.1) can also be written in the form

$$N(\lambda) - \frac{|\Omega|_n}{(2\pi)^n}(n-1)|B^n|_n\lambda^{n/2} = o(\lambda^{n/2}) \qquad as \quad \lambda \to \infty, \tag{0.2}$$

which prompts us for the improvement of the estimate of the remainder.

Actually, Babenko [1] showed that, for $n = 3$ and smooth $\partial\Omega$, the $o(\lambda^{n/2})$ in (0.2) can be improved to $O(\frac{\lambda^{n/2}}{\ln \lambda})$, and Kozhevnikov [7] even got the estimate $O(\lambda^{(n-1)/2})$ for the remainder if $\partial\Omega$ is assumed to be infinitely smooth.

In [2] we conjectured, again in the case when Ω is assumed to be Lipschitz, that the little o estimate in (0.2) could be replaced by $O(\lambda^{(n-1/5)/2})$ (and we even suggested that $O(\lambda^{(n-1)/2})$ might also hold). It is a fact, however, that Levendorskiĭ [9] had already proved that the estimate $O(\lambda^{(n-\delta)/2})$, for any $\delta \in (0, 1/2)$, holds in this case.

As a by-product of the results proved in the present work, we in fact show that, in the case Ω is Lipschitz, the remainder in (0.2) is a $O(\lambda^{(n-1)/2}\ln\lambda)$.

We prove more than this, however.

First of all, we obtain (0.1) for any bounded open non-empty subset Ω of $\mathbb{R}^n$ such that $|\partial\Omega|_n = 0$ (see (6.5)). Secondly, we show that the remainder in (0.2) is $O(\lambda^{D/2})$ whenever $\partial\Omega$ has (inner) Minkowski dimension equal to $D \in (n-1, n]$ and its D-dimensional upper (inner) Minkowski content is finite (cf. Corollary 1.3). Actually, we even show something broader than this, as we also get a similar result for some more general dimension functions used instead of the standard power dimension function associated with D (cf. Theorem 1.2).

The approach used is the same as Métivier's [10] — as was the case in [8] — for elliptic operators, which is well adapted to the situation where the boundary of Ω is extremely irregular.

As will be apparent in section 1, when defining the Stokes operator we take the point of view of using the space

$$\{u \in (H_0^1(\Omega))^n : \operatorname{div} u = 0\}$$

for the V-space that appears in the literature on the Navier-Stokes equations (cf. [12, 4], for example). It is known (cf. [12, p.23]) that this might not be the same as considering the closure of $\{u \in (C_0^\infty(\Omega))^n : \operatorname{div} u = 0\}$ in $(H_0^1(\Omega))^n$ for the V-space, mainly if $\partial\Omega$ is not smooth — which is the case we are more interested in. However, it is not difficult to see that our results are blind to this distinction — the same kind of arguments could be applied were we to consider the other setting (in section 4 we don't even need to distinguish between the two, because there we deal only with the case when Ω is a n-cube).

Since there are some general procedures that we repeat several times along the text, we would like to make the following conventions.

Whenever we consider the closure A of a subset in a Hilbert space B, the inner product to be considered in A is, if nothing is said to the contrary, the one naturally inherited (i.e., by restriction) from B. Also, whenever we consider product spaces of Hilbert spaces, the inner product to be considered in the product space is, if nothing is said to the contrary, the one we can build naturally (cf., e.g., (1.4), which defines $(\cdot,\cdot)_{H(\Omega)}$).

As to the notation, $(\cdot,\cdot)_A$ and $\|\cdot\|_A$ stand, respectively, for the inner product and norm in A. Also, the letter c, possibly with subscripts and/or superscripts, is used for a *positive constant*, the precise value of which is unimportant for us. And, though we have made an effort to use different c's in neighbouring formulae, the use of the same letter c in two of them does not necessarily mean that the two c's represent the same value.

1 Setting of the problem and main results

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space (with $n \geq 2$) and let Ω stand for an arbitrary bounded open non-empty subset of $\mathbb{R}^n$.

We shall use the standard notation $L_2(\Omega)$ for the space of (equivalence classes of) complex measurable functions on Ω which are square-integrable with respect to the Lebesgue σ-field and measure. Moreover, $H^1(\Omega)$ will stand for the Sobolev space consisting of the functions of $L_2(\Omega)$ which have first order weak partial derivatives also in $L_2(\Omega)$.

These spaces are endowed with the usual Hilbert structures, by means of the usual inner products, namely

$$(u,v)_{L_2(\Omega)} \equiv \int_\Omega u(x)\overline{v(x)}dx, \tag{1.1}$$

$$(u,v)_{H^1(\Omega)} \equiv (u,v)_{L_2(\Omega)} + \sum_{j=1}^{n} \int_\Omega \frac{\partial u}{\partial x_j}(x)\overline{\frac{\partial v}{\partial x_j}(x)}dx. \tag{1.2}$$

Also as usual, $H_0^1(\Omega)$ will denote the closure of $C_0^\infty(\Omega)$ (the space of infinitely continuously differentiable complex functions with compact support on Ω) in $H^1(\Omega)$.

The Stokes operator arises when we consider the variational form of Stokes' problem. In order to define it, we need thus to consider the following sesquilinear form a:

$$a(u,v) \equiv \sum_{i=1}^{n}\sum_{j=1}^{n} \int_\Omega \frac{\partial u_i}{\partial x_j}(x)\overline{\frac{\partial v_i}{\partial x_j}(x)}dx, \tag{1.3}$$

for $u \equiv (u_i)_{i=1}^n, v \equiv (v_i)_{i=1}^n \in (H^1(\Omega))^n$, where $(H^1(\Omega))^n$ is the product of n copies of $H^1(\Omega)$.

We note that $(H^1(\Omega))^n$, which will be denoted more simply by $H(\Omega)$, is a Hilbert space for the natural inner-product we can define in a product space of

Hilbert spaces:

$$(u,v)_{H(\Omega)} \equiv \sum_{i=1}^{n} (u_i, v_i)_{H^1(\Omega)}. \tag{1.4}$$

Analogously, $L(\Omega)$ will denote the Hilbert product space $(L_2(\Omega))^n$ with the natural inner product given by

$$(u,v)_{L(\Omega)} \equiv \sum_{i=1}^{n} (u_i, v_i)_{L_2(\Omega)}.$$

The form a in $H(\Omega)$ is obviously Hermitian, continuous and coercive with respect to $L(\Omega)$ (cf. footnotes 1 and 2 in section 2), as we have, for all $u, v \in H(\Omega)$,

$$a(u,v) = \overline{a(v,u)}, \tag{1.5}$$

$$|a(u,v)| \leq \|u\|_{H(\Omega)} \|v\|_{H(\Omega)} \tag{1.6}$$

$$\|u\|^2_{H(\Omega)} - \|u\|^2_{L(\Omega)} = a(u,u). \tag{1.7}$$

We need still another space:

$$V_0(\Omega) \equiv \{u \in H_0(\Omega) : \operatorname{div} u = 0\}, \tag{1.8}$$

where $H_0(\Omega) \equiv (H_0^1(\Omega))^n$ and $\operatorname{div} u$ stands for the *divergence* of u (i.e., $\operatorname{div} u = \sum_{i=1}^{n} \partial u_i / \partial x_i$).

Clearly, both $H_0(\Omega)$ and $V_0(\Omega)$ are complete subspaces of $H(\Omega)$ — and so are Hilbert spaces with respect to the restriction to those spaces of the inner product in $H(\Omega)$.

In view of what was pointed out above, we have that the form a is also Hermitian, continuous and coercive with respect to $L(\Omega)$ in $H_0(\Omega)$ and $V_0(\Omega)$ — actually, (1.5) to (1.7) hold true in these spaces.

Since

$$\|u\|_{L(\Omega)} \leq \|u\|_{H(\Omega)}, \quad \text{for all } u \in H(\Omega), \tag{1.9}$$

we can say that all the three spaces $V_0(\Omega)$, $H_0(\Omega)$ and $H(\Omega)$ are continuously embedded in $L(\Omega)$, so that the following are *variational triplets* (cf. section 2):

$$(V_0(\Omega), L(\Omega), a), \quad (H_0(\Omega), L(\Omega), a), \quad (H(\Omega), L(\Omega), a). \tag{1.10}$$

(Of course, in each case a is restricted to pairs of functions belonging to the first element of the triplet, though we always use the same letter a — such a convention will be in force throughout).

For technical reasons, one temporarily needs one further space, namely $L_0(\Omega)$, which is defined as the closure of $V_0(\Omega)$ in $L(\Omega)$. It is endowed with the Hilbert structure of $L(\Omega)$ (becoming itself a Hilbert space), so that

$$(V_0(\Omega), L_0(\Omega), a)$$

is also a variational triplet. The point is that now we have $V_0(\Omega)$ densely embedded in the second space of the triplet, this allowing us to associate with the form a in $V_0(\Omega)$ a lower semi-bounded self-adjoint operator A in $L_0(\Omega)$ by means of the Lax-Milgram lemma.

It is this A that is called the *Stokes operator*.

Note now that the embedding $V_0(\Omega) \to L_0(\Omega)$ is compact, as it can be obtained by means of the diagram

$$\begin{array}{ccc} V_0(\Omega) & \to & L_0(\Omega) \\ \downarrow & & \uparrow \\ H_0(\Omega) & \to & L(\Omega), \end{array}$$

where $V_0(\Omega) \to H_0(\Omega)$ and $H_0(\Omega) \to L(\Omega)$ are the natural embeddings and $L(\Omega) \to L_0(\Omega)$ is the orthogonal projection; since the lower embedding of the diagram is compact (recall that we are assuming Ω bounded), the proof for our claim of the compacity of $V_0(\Omega) \to L_0(\Omega)$ follows immediately.

This fact implies that the spectrum of the Stokes operator is formed of eigenvalues alone, and that these can be written in a sequence $(\lambda_k)_{k\in\mathbf{N}}$ obeying the following:

$$\lambda_1 \leq \lambda_2 \leq \ldots \to +\infty.$$

We suppose (as is usual) that the λ_k's in this sequence appear repeated according to multiplicity, so that a corresponding sequence of orthonormal eigenfunctions constitutes a basis for the space $L_0(\Omega)$.

The main objective of this paper is to study the asymptotic behaviour of such a sequence $(\lambda_k)_{k\in\mathbf{N}}$ when k goes to infinity. Also as usual, we adopt the point of view of the *counting function*

$$N(\lambda) = \#\{k \in \mathbb{N} : \lambda_k \leq \lambda\} \tag{1.11}$$

to accomplish that goal.

In order to state our main result, we need to say what is meant by functions of class $\mathcal{H}$.

Definition **1.1** *A function $h : [c, +\infty) \to \mathbb{R}$, with $c > 0$, is said to be of class $\mathcal{H}$ if it is strictly positive, differentiable and if*

$$\lim_{x\to\infty} \frac{xh'(x)}{h(x)} = 0.$$

In what follows, the following notations will be consistently used: $|\cdot|_n$ for Lebesgue measure in $\mathbb{R}^n$; B^n for the Euclidean unit ball in $\mathbb{R}^n$; for $\varepsilon > 0$,

$$(\partial\Omega)_\varepsilon \equiv \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \varepsilon\} \tag{1.12}$$

And now for our main result:

THEOREM **1.2** *Let Ω be a bounded open non-empty subset of $\mathbb{R}^n$ and $d > n-1$. Assume there exists a function $f(x) = x^{d/2}h(x)$, for some $h \in \mathcal{H}$, such that*

$$\limsup_{\varepsilon\to 0^+} \frac{|(\partial\Omega)_\varepsilon|_n}{\varepsilon^n f(\varepsilon^{-2})} < +\infty.$$

Then

$$N(\lambda) = \frac{|\Omega|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2} + O(f(\lambda)) \quad \textit{as} \;\; \lambda \to \infty. \tag{1.13}$$

If, on the other hand, $d = n-1$ and $h \equiv 1$, and if we make the same assumption as before for the corresponding lim sup, *the same result holds with the last term replaced by $O(\lambda^{(n-1)/2}\ln\lambda)$.*

Remark. The second part of the theorem applies, in particular, to the case when Ω is Lipschitz.

COROLLARY **1.3** *Let Ω be a bounded open non-empty subset of $\mathbb{R}^n$ such that $\partial\Omega$ has (inner) Minkowski dimension $D \in (n-1, n]$. Assume that*

$$\limsup_{\varepsilon\to 0^+} \frac{|(\partial\Omega)_\varepsilon|_n}{\varepsilon^{n-D}} < +\infty.$$

Then

$$N(\lambda) = \frac{|\Omega|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2} + O(\lambda^{D/2}) \quad \textit{as} \;\; \lambda \to \infty. \tag{1.14}$$

Remark. We note once and for all that this corollary is an immediate consequence of the theorem: one just has to read the latter with d and $h(x)$ replaced by D and 1 respectively, for the assertion that $\partial\Omega$ has (inner) Minkowski dimension D just means that

$$D = \inf\{d \geq 0 : \limsup_{\varepsilon\to 0^+} \frac{|(\partial\Omega)_\varepsilon|_n}{\varepsilon^{n-d}} < +\infty\}.$$

(and note that such a D is known to belong necessarily to $[n-1, n]$ — cf. [8, p.475]).

From this and the theorem it is also clear that if the boundary of the bounded open non-empty subset Ω of $\mathbb{R}^n$ has Minkowski dimension D then, with no further assumptions, (1.14) holds with d instead of D for all $d > D$.

2 Preliminaries

It is not necessary for us to work with the above space $L_0(\Omega)$: we have (following [10]) alternative means to characterize $N(\lambda)$.

We recall that a *variational triplet* is a triplet (V, H, a) where V and H are complex Hilbert spaces such that V is continuously embedded in H, and where a is a Hermitian, continuous and coercive[1] (with respect to H) sesquilinear form in V. One then defines, for all $\lambda \in \mathbb{R}$,

$$N(\lambda, V, H, a) \equiv \inf \operatorname{codim}_V(E), \tag{2.1}$$

where the infimum is taken over all closed subspaces E of V such that the form $a - \lambda(\cdot,\cdot)_H$ is strongly coercive[2] in E ($\operatorname{codim}_V(E)$ denotes the (finite or infinite) co-dimension of E in V).

The relation of this concept with $N(\lambda)$ comes from the following [10, p.143]

Proposition **2.1** *Let (V, H, a) be a variational triplet with V densely and compactly embedded in H. Then, for all $\lambda \in \mathbb{R}$,*

$$N(\lambda, V, H, a) = \#\{k \in \mathbb{N} : \lambda_k \leq \lambda\},$$

[1] Coercivity of a in V with respect to H means that there is a $\lambda_0 \in \mathbb{R}$ such that $a + \lambda_0(\cdot,\cdot)_H$ is strongly coercive in V.

[2] A form b in V is said to be strongly coercive in a subspace E of V if there is a $m > 0$ such that the relation $m\|u\|_V^2 \leq b(u,u)$ holds for all $u \in E$.

where $(\lambda_k)_{k\in\mathbf{N}}$ *is the sequence of eigenvalues of the operator in* H *associated with* a *by means of the Lax-Milgram lemma.*

Using the notations of the preceding section, if we apply this to the case when $V = V_0(\Omega)$, $H = L_0(\Omega)$ and a is the form given by (1.3), we get

$$N(\lambda) = N(\lambda, V_0(\Omega), L_0(\Omega), a), \qquad \lambda \in \mathrm{I\!R}, \tag{2.2}$$

which gives a characterization of the counting function for the Stokes operator (originally defined by (1.11)).

Remark. We still call $N(\cdot, V, H, a)$ a *counting function*, even if the hypotheses of the last proposition are not verified.

From the definition of $N(\lambda, V, H, a)$ it follows that

$$N(\lambda) = N(\lambda, V_0(\Omega), L(\Omega), a), \qquad \lambda \in \mathrm{I\!R}. \tag{2.3}$$

This is what we had in mind when stating that the space $L_0(\Omega)$ was not needed in what follows — for the purpose of studying the asymptotics of the eigenvalues of the Stokes operator, we can work with the variational triplet $(V_0(\Omega), L(\Omega), a)$ instead of $(V_0(\Omega), L_0(\Omega), a)$.

2.1 A special variational triplet

Let Ω be an open non-empty subset of $\mathrm{I\!R}^n$ and V a Hilbert space continuously embedded in $L(\Omega)$. Let ω be an open non-empty subset of Ω and define the space

$$\mathcal{V} = \{v|_\omega : v \in V\}.$$

It is known (see, for example, [10, p.146]) that $\mathcal{V}$ is a Hilbert space when endowed with the inner product corresponding to the norm defined by

$$\|u\|_{\mathcal{V}} \equiv \inf_{\substack{v\in V \\ v|_\omega = u}} \|v\|_V, \qquad \forall u \in \mathcal{V}. \tag{2.4}$$

Consider the sesquilinear form defined by

$$(u, u')_{\mathcal{V}} \equiv (v - Pv, v' - Pv')_V, \ \ \forall u, u' \in \mathcal{V},$$

where v, v' are any elements of V satisfying the identities $v|_\omega = u$, $v'|_\omega = u'$ and P is the orthogonal projection onto the subspace $\{v \in V : v|_\omega = 0\}$.

It is a nice exercise to check that this form is in fact an inner product and that the norm corresponding to it is precisely given by (2.4). We can also say that

$$(\mathcal{V}, L(\omega), (\cdot,\cdot)_{\mathcal{V}})$$

is a variational triplet.

3 The method

Let Ω be as in section 1: an arbitrary bounded open non-empty subset of $\mathbb{R}^n$ (the notation we are going to use in this section is indeed consistent with the one introduced in section 1).

In view of our discussion in the preceding section, we are trying to prove an asymptotic formula like (1.13) for

$$N(\lambda, V_0(\Omega), L(\Omega), a)$$

— as this is the same as $N(\lambda)$ of (1.11).

For each $r \in \mathbb{N}_0$ consider the tessellation $\{J^r_\nu : \nu \in \mathbb{Z}^n\}$ of $\mathbb{R}^n$ by the open n-dimensional cubes $J^r_\nu \equiv \prod_{i=1}^n]2^{-r}\nu_i, 2^{-r}(\nu_i+1)[$. We define, by induction on r, the following sets A_r, Ω_r and ω_r (cf. also Figure 1):

$$r = 0: \qquad A_0 \equiv \{\nu \in \mathbb{Z}^n : \overline{J^0_\nu} \subset \Omega\};$$

$$\Omega_0 \equiv \cup_{\nu \in A_0} J^0_\nu;\ \omega_0 \equiv \Omega \setminus \overline{\Omega_0};$$

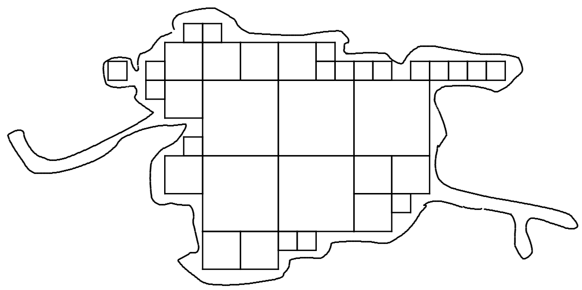

Figure 1: Example of Ω and Ω_r in the case $n = 2$.

$$r \in \mathbb{N}: \qquad A_r \equiv \{\nu \in \mathbb{Z}^n : \overline{J_\nu^r} \subset \Omega \wedge J_\nu^r \cap \Omega_{r-1} = \emptyset\};$$

$$\Omega_r \equiv \Omega_{r-1} \cup (\cup_{\nu \in A_r} J_\nu^r)\,;\ \omega_r \equiv \Omega \setminus \overline{\Omega_r}$$

In what follows we consider $r \geq r_0$, where r_0 is the smallest number $r \in \mathbb{N}_0$ such that $\Omega_r \neq \emptyset$.

With an eye on formula (1.13), which we want ultimately to prove, we can write, for all $\lambda \in \mathbb{R}$ and all integers $r \geq r_0$,

$$\begin{aligned} & N(\lambda, V_0(\Omega), L(\Omega), a) - \frac{|\Omega|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2} \\ \geq\ & \sum_{\rho,\nu} N(\lambda, V_0(J_\nu^\rho), L(J_\nu^\rho), a_\nu^\rho) - \sum_{\rho,\nu} \frac{|J_\nu^\rho|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2} \\ & - \frac{|\omega_r|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2} \\ \geq\ & \sum_{\rho,\nu} \left(N(\lambda, V_0(J_\nu^\rho), L(J_\nu^\rho), a_\nu^\rho) - \frac{|J_\nu^\rho|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2} \right) \\ & - \frac{n-1}{(2\pi)^n}|B^n|_n |(\partial\Omega)_{(\sqrt{n}+1)2^{-r}}|_n \lambda^{n/2}, \end{aligned} \tag{3.1}$$

where a_ν^ρ stands for the form in $V_0(J_\nu^\rho)$ given by expression (1.3) but with Ω replaced by J_ν^ρ and the summation $\sum_{\rho,\nu}$ runs over all $\rho \in [r_0, r]$ and $\nu \in A_\rho$. We remark that the last inequality in (3.1) follows from the — easy to prove — inclusion

$$\omega_r \subset (\partial\Omega)_{(\sqrt{n}+1)2^{-r}} \tag{3.2}$$

— recall the definition of $(\partial\Omega)_\varepsilon$ in (1.12); as to the first inequality, it follows as in the case of the Dirichlet Laplacian, using now the abstract setting developed in [10, Ch. II], in particular [10, Lem. 2.1, Lem. 2.5, Prop. 2.8].

The same abstract setting developed in [10, Ch. II], in particular [10, Prop. 2.7, Lem. 2.1, Prop. 2.8], allows us to get an inequality opposite to that of (3.1), namely, for all $\lambda \in \mathbb{R}$ and all integers $r \geq r_0$,

$$\begin{aligned} & N(\lambda, V_0(\Omega), L(\Omega), a) - \frac{|\Omega|_n}{(2\pi)^n}(n-1)|B^n|_n\lambda^{n/2} \\ \leq \; & \sum_{\rho,\nu}\left(N(\lambda, V_0(J_\nu^\rho), L(J_\nu^\rho), a_\nu^\rho) - \frac{|J_\nu^\rho|_n}{(2\pi)^n}(n-1)|B^n|_n\lambda^{n/2}\right) \\ & + N(\lambda, Z_\lambda^r(\Omega), L(\Omega), a), \end{aligned} \tag{3.3}$$

where

$$Z_\lambda^r(\Omega) \equiv \{u \in V_0(\Omega) : \forall v \in V_0(\Omega_r),\; a(u,\tilde{v}) = \lambda(u,\tilde{v})_{L(\Omega)}\}, \tag{3.4}$$

with $\tilde{v}$ denoting the extension of v by 0 outside Ω_r.

This is the exact counterpart of what is done to get a corresponding inequality in the case of the Dirichlet Laplacian (cf. [10] or [8]).

From (3.1) and (3.3) we see the need to control

$$N(\lambda, V_0(J_\nu^\rho), L(J_\nu^\rho), a_\nu^\rho)$$

for large values of λ (which we shall be doing in section 4) and the need to estimate

$$N(\lambda, Z_\lambda^r(\Omega), L(\Omega), a)$$

from above when λ goes to infinity.

Arguing as in [10, Lem. 5.8], one obtains

$$
\begin{aligned}
N(\lambda, Z_\lambda^r(\Omega), L(\Omega), a) \quad \leq \quad & N(2(\lambda+1), \mathcal{V}_0(\omega_r), L(\omega_r), (\cdot,\cdot)_{\mathcal{V}_0(\omega_r)}) \\
& + \sum_{\rho,\nu} N(2(\lambda+1), Z_\lambda(J_\nu^\rho), L(J_\nu^\rho), (\cdot,\cdot)_{H(J_\nu^\rho)}),
\end{aligned} \tag{3.5}
$$

where $\mathcal{V}_0(\omega_r)$ — the set of restrictions to ω_r of the elements of $V_0(\Omega)$ — is made a Hilbert space by means of the procedure described in subsection 2.1 for the space $\mathcal{V}$ and, for each integer $\rho \geq r_0$ and each $\nu \in A_\rho$, the space $Z_\lambda(J_\nu^\rho)$ is defined by

$$
Z_\lambda(J_\nu^\rho) \equiv \{u \in H(J_\nu^\rho) : \operatorname{div} u = 0 \ \wedge \ \forall v \in V_0(J_\nu^\rho), \ a_\nu^\rho(u,v) = \lambda(u,v)_{L(J_\nu^\rho)}\}. \tag{3.6}
$$

The question of estimating $N(\lambda, Z_\lambda^r(\Omega), L(\Omega), a)$ from above is then reduced to estimating from above the counting functions

$$
N(2(\lambda+1), Z_\lambda(J_\nu^\rho), L(J_\nu^\rho), (\cdot,\cdot)_{H(J_\nu^\rho)})
$$

and

$$
N(2(\lambda+1), \mathcal{V}_0(\omega_r), L(\omega_r), (\cdot,\cdot)_{\mathcal{V}_0(\omega_r)})
$$

This will be done in section 5 for the latter, by means of what is called, for obvious reasons, an estimate *near the boundary* (of Ω). As to the other two types of counting function, the underlying sets of which are the n-cubes J_ν^ρ, they will be dealt with in section 4.

4 Estimates for cubes

4.1 The problem when Ω is a cube

We will first reduce the problem of controlling

$$
N(\lambda, V_0(J_\nu^\rho), L(J_\nu^\rho), a_\nu^\rho),
$$

both from below and from above, to similar problems involving periodic functions. We use the same method as was applied in section 3 in order to reduce part of our original problem to the problem now under consideration.

We need first to introduce some notations.

The letter J will stand for an arbitrary n-cube in $\mathbb{R}^n$, and δ for its side length. We define, moreover, the spaces

$$C^\infty_\#(J) \equiv \{f|_J : f \in C^\infty(\mathbb{R}^n) \text{ and is periodic of period } \delta \text{ in each coordinate}\},$$

$$H^1_\#(J) \equiv \text{“closure of } C^\infty_\#(J) \text{ in } H^1(J)\text{”}$$

$$H_\#(J) \equiv (H^1_\#(J))^n;$$

$$V_\#(J) \equiv \{u \in H_\#(J) : \operatorname{div} u = 0\}.$$

It is clear that $V_\#(J)$ is a Hilbert space for the inner product inherited from that in $H_\#(J)$ (which, in turn, is inherited from $H(J)$). Moreover

$$(V_\#(J), L(J), a_J)$$

is a variational triplet, where a_J is given by formula (1.3) but with Ω replaced by J.

Since $V_0(J)$ is a closed subspace of $V_\#(J)$, we can apply [10, Lem. 2.5] and write, for all $\lambda \in \mathbb{R}$,

$$\begin{aligned} & N(\lambda, V_0(J), L(J), a_J) - \frac{|J|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2} \\ \leq \quad & N(\lambda, V_\#(J), L(J), a_J) - \frac{|J|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2}. \end{aligned} \tag{4.1}$$

If we use, on the other hand, [10, Prop. 2.7], we get, for all $\lambda \in \mathbb{R}$,

$$N(\lambda, V_0(J), L(J), a_J) - \frac{|J|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2}$$

$$\geq \quad N(\lambda, V_{\#}(J), L(J), a_J) - \frac{|J|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2}$$
$$-N(\lambda, Z_\lambda^{\#}(J), L(J), a_J), \tag{4.2}$$

where

$$Z_\lambda^{\#}(J) \equiv \{u \in V_{\#}(J) : \forall v \in V_0(J), a_J(u,v) = \lambda(u,v)_{L(J)}\}. \tag{4.3}$$

We see from (4.1) and (4.2) that we have reduced the problem mentioned in the starting of this subsection to the controlling of

$$N(\lambda, V_{\#}(J), L(J), a_J),$$

both from below and from above, and to obtaining upper estimates for

$$N(\lambda, Z_\lambda^{\#}(J), L(J), a_J).$$

Observe that we can easily relate this latter counting function with one we already met before, namely one mentioned at the end of section 3 and which we also need to estimate. In fact, recalling (3.6), we can write

$$Z_\lambda^{\#}(J) = Z_\lambda(J) \cap H_{\#}(J),$$

so that $Z_\lambda^{\#}(J)$ is a closed subspace of $Z_\lambda(J)$, and [10, Lem. 2.5] applies to yield, for all $\lambda \in \mathbb{R}$,

$$N(\lambda, Z_\lambda^{\#}(J), L(J), a_J) \leq N(\lambda, Z_\lambda(J), L(J), a_J);$$

since $(\cdot,\cdot)_{H(J)} = (\cdot,\cdot)_{L(J)} + a_J(\cdot,\cdot)$ (cf. (1.7) for a particular case) we can finally write, for all $\lambda \in \mathbb{R}$,

$$N(\lambda, Z_\lambda^{\#}(J), L(J), a_J) \leq N(\lambda+1, Z_\lambda(J), L(J), (\cdot,\cdot)_{H(J)}). \tag{4.4}$$

Our problem thus reduces to controlling, both from below and from above,

$$N(\lambda, V_\#(J), L(J), a_J)$$

for big values of λ and to obtain suitable upper estimates for

$$N(\mu, Z_\lambda(J), L(J), (\cdot,\cdot)_{H(J)}),$$

at least for $\mu = \lambda + 1$ and $\mu = 2(\lambda + 1)$. This is the same approach as the one used for elliptic problems in [10].

4.2 Scaling and translation properties

Instead of dealing with an arbitrary n-cube J, it is a standard argument to restrict attention to one particular n-cube and afterwards invoke a scaling and translation argument to get the corresponding results in the general case.

We summarize the relevant results in our context.

Denoting by $d \in \mathbb{R}^n$ the centre of the n-cube J with side length $\delta > 0$, and by Q the n-cube $]-\pi,\pi[^n$, consider the application

$$T : L(J) \to L(Q)$$

given by $(Tu)(x) = \left(\frac{\delta}{2\pi}\right)^{n/2} u(\frac{\delta}{2\pi}x + d)$. It is easy to see that T is unitary and that $H(J)$ and $H(Q)$ are linearly homeomorphic under this map. Also,

$$T(V_\#(J)) = V_\#(Q),$$

$$T(Z_\lambda(J)) = Z_{(\delta/(2\pi))^2\lambda}(Q), \quad \forall \lambda \in \mathbb{R} \text{ , and}$$

$$a_Q(u,v) = \left(\frac{\delta}{2\pi}\right)^2 a_J(T^{-1}u, T^{-1}v), \quad \forall u,v \in H(Q),$$

so that

$$N(\lambda, V_\#(J), L(J), a_J) = N((\delta/(2\pi))^2\lambda, V_\#(Q), L(Q), a_Q), \tag{4.5}$$

for all $\lambda \in \mathbb{R}$, and

$$N(\mu, Z_\lambda(J), L(J), (\cdot,\cdot)_{H(J)}) \\ = N((\delta/(2\pi))^2(\mu-1)+1, Z_{(\delta/(2\pi))^2\lambda}(Q), L(Q), (\cdot,\cdot)_{H(Q)}), \tag{4.6}$$

for all $\mu \in \mathbb{R}$.

4.3 Controlling $N(\lambda, V_\#(J), L(J), a_J)$

This can be recovered from the result in [2, p.113], which deals with a similar problem in a much more general context, but we shall briefly sketch a simpler approach for our present concrete situation (besides, the considerations we are going to make here in this connection will be useful later on).

In accordance with the preceding subsection, we consider first the case $J = Q =]-\pi, \pi[^n$.

Note first that the counting function now under discussion is the same as

$$N(\lambda, V_\#(Q), L_\#(Q), a_Q),$$

where $L_\#(Q)$ stands for the closure of $V_\#(Q)$ in $L(Q)$. Then, as in section 1, we can associate with the form a_Q in $V_\#(Q)$ a lower semi-bounded self-adjoint operator A_Q in $L_\#(Q)$ by means of the Lax-Milgram lemma. The compactness of the natural embedding $H^1(Q) \to L_2(Q)$ implies, moreover, that our counting function is nothing more than

$$\#\{k \in \mathbb{N} : \lambda_k \leq \lambda\}$$

where $(\lambda_k)_{k\in\mathbf{N}}$ is now the sequence of eigenvalues of A_Q (cf. what we have done in section 1 with respect to the form a and the operator A).

In order to get an impression of the structure of this sequence of eigenvalues, recall (see, e.g., [2, p.32] or [5, pp.175-176]) that the domain of A_Q is

$$\mathcal{D}(A_Q) \equiv \{u \in V_\#(Q) : a_Q(u,w) = (v,w)_{L(Q)} \text{ for some } v \in L_\#(Q) \text{ and all } w \in V_\#(Q)\};$$

and that the v in this definition is precisely the image of u under A_Q.

We proceed by using the Fourier series representation for the functions involved. We know that

$$\{\varphi_k\}_{k\in\mathbb{Z}^n} \equiv \{(2\pi)^{-n/2}\exp(ik.x)\}_{k\in\mathbb{Z}^n}$$

is a complete orthonormal system in $L_2(Q)$, where $x \in \mathbb{R}^n$ stands for the variable of φ_k and $k.x$ denotes the inner product of k and x in $\mathbb{R}^n$. From this we get the *representation*

$$u = \sum_{k\in\mathbb{Z}^n} \varphi_k\, \hat{u}(k)$$

in $L(Q)$ for any $u \equiv (u_j)_{j=1}^n$ in this space, where $\hat{u}(k) = (\widehat{u_j}(k))_{j=1}^n$, $\widehat{u_j}(k) = (u_j, \varphi_k)_{L_2(Q)}$, and $\varphi_k\, \hat{u}(k) \equiv (\widehat{u_j}(k)\varphi_k)_{j=1}^n$; and if $u \in H_\#(Q)$ then the above representation is even valid in the topology of $H(Q)$.

It is not difficult to see that

$$V_\#(Q) = \{u \in H_\#(Q) : \forall k \in \mathbb{Z}^n,\ \hat{u}(k).k = 0\}$$

and

$$L_\#(Q) \subset \{u \in L(Q) : \forall k \in \mathbb{Z}^n,\ \hat{u}(k).k = 0\} \equiv L_\perp(Q),$$

where $\hat{u}(k).k$ denotes the inner product of $\hat{u}(k)$ and k in $\mathbb{C}^n$. Also, denoting by $\{e_j\}_{j=1}^n$ the canonical basis of $\mathbb{C}^n$ and by $\{e_j^k\}_{j=1}^{n-1}$ an orthonormal basis of $< k >^\perp$ in $\mathbb{C}^n$, $k \in \mathbb{Z}^n \setminus \{0\}$, the system

$$\{\varphi_0 e_j\}_{j=1}^n \cup \{\varphi_k e_j^k\}_{\substack{k\in\mathbb{Z}^n\setminus\{0\} \\ j\in\{1,\dots,n-1\}}} \tag{4.7}$$

is complete orthonormal in the closed subspace $L_\perp(Q)$ of $L(Q)$ — in particular, we have $L_\perp(Q) = L_\#(Q)$; on the other hand, the system

$$\{\varphi_k e_j\}_{\substack{k\in\mathbb{Z}^n \\ j\in\{1,\dots,n\}}} \tag{4.8}$$

is complete orthonormal in $L(Q)$.

Now it is an easy task to show that, with u any of the elements of the system (4.7) and w any of the elements of the system (4.8), the following holds true:

$$a_Q(u,w) = (|k|^2 u, w)_{L(Q)}, \tag{4.9}$$

where $k \in \mathbb{Z}^n$ denotes the index of φ in u. And from this it is an easy matter to check that (4.9) indeed holds true for all $w \in H_\#(Q)$.

It is clear that this is more than enough to prove that, given any u in the system (4.7),

$$u \in \mathcal{D}(A_Q) \quad \text{and} \quad A_Q u = |k|^2 u, \tag{4.10}$$

where $k \in \mathbb{Z}^n$ has the same meaning as in (4.9) above.

The self-adjointness of A_Q and the completeness of the system (4.7) in $L_\#(Q)$ implies now that the sequence of values $|k|^2$ (disposed in non-decreasing order) corresponding to each and every eigenvector in (4.7) is precisely the sequence $(\lambda_k)_{k\in\mathbf{N}}$ of the eigenvalues of A_Q.

As a result (cf. Proposition 2.1 for the relation between the counting function under discussion and the eigenvalues just mentioned)

$$N(\lambda, V_\#(Q), L(Q), a_Q) = n + (n-1).\#\{k \in \mathbb{Z}^n \setminus \{0\} : |k|^2 \le \lambda\} \tag{4.11}$$

for $\lambda \ge 0$ (it equals 0 otherwise). From this formula, some volume estimates can be used (as in the case of the Laplacian) in order to get the desired result (cf. also

[4, pp.43-44]). We have

$$\left| N(\lambda, V_{\#}(Q), L(Q), a_Q) - \frac{|Q|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2} \right| \leq c_n \lambda^{(n-1)/2}$$

for all $\lambda \geq 1$, where c_n depends only on n.

We can get rid of this restriction on λ (which can be a bit annoying when scaling) in the following sense: since our counting function is monotone increasing in λ, then we can write

$$\left| N(\lambda, V_{\#}(Q), L(Q), a_Q) - (n-1)|B^n|_n \lambda^{n/2} \right| \leq c_n (1 + \lambda^{(n-1)/2})$$

for all $\lambda \geq 0$, possibly after redefining c_n (still depending only on n).

From this we finally get for any J, using the scaling and translation argument in (4.5),

$$\begin{aligned} \left| N(\lambda, V_{\#}(J), L(J), a_J) - \frac{|J|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2} \right| & \\ \leq c_n (1 + (\delta/(2\pi))^{n-1} \lambda^{(n-1)/2}) & \end{aligned} \tag{4.12}$$

for all $\lambda \geq 0$.

4.4 Estimating $N(\mu, Z_\lambda(J), L(J), (\cdot,\cdot)_{H(J)})$

We try to follow closely the proof of [10, Prop. 4.1.i].

Again, according to subsection 4.2, we consider first the case $J = Q =]-\pi, \pi[^n$.

For fixed $\mu, \lambda \geq 0$, the idea is to find, in accordance with the definition (2.1) of counting function, a closed subspace E of $Z_\lambda(Q)$ such that

$$\exists \varepsilon > 0 : \forall u \in E, \ (u,u)_{H(Q)} - \mu(u,u)_{L(Q)} \geq \varepsilon \|u\|^2_{L(Q)} \tag{4.13}$$

(cf. also [10, Lem. 2.1]) and, moreover, with $\mathrm{codim}_{Z_\lambda(Q)}(E)$ bounded above by something of the order of $\mu^{(n-1)/2} + \lambda^{(n-1)/2}$.

Using $\varphi_k e_j$, for $k \in \mathbb{Z}^n$ and $j \in \{1, \ldots, n\}$, in the same sense as in (4.8), we will first prove that the closed subspace

$$\begin{aligned} Z \quad \equiv \quad & \{u \in Z_\lambda(Q) : (u, \varphi_k e_j)_{L(Q)} = 0 \text{ for all } k \in \mathbb{Z}^n \\ & \text{such that } |k|^2 \leq \nu \text{ and all } j \in \{1, \ldots, n\}\}, \end{aligned}$$

of $Z_\lambda(Q)$ has co-dimension of the required order, provided $\nu(\geq 0)$ is chosen in a suitable way.

In what follows we shall write $\varphi_k e_j^k$, for $k \in \mathbb{Z}^n \setminus \{0\}$ and $j \in \{1, \ldots, n-1\}$, in the same sense as in (4.7), and also $\varphi_0 e_j^0$, for $j \in \{1, \ldots, n\}$, as meaning the same as $\varphi_0 e_j$ in (4.7). We also remark that, to take advantage of the definition of $Z_\lambda(Q)$, one must deal with functions the divergence of which is zero. This is why we are going to consider now the system (4.7) and not the system (4.8). This is a point that doesn't show up in the case of elliptic problems. We have a price to pay for it, however, as (4.7) does not span $L(Q)$: later on one must study also what happens with the system $\{\varphi_k k\}_{k \in \mathbb{Z}^n}$ in order to recover the picture in the whole space $L(Q)$.

Note that, given $u \in Z_\lambda(Q)$ and $k \in \mathbb{Z}^n$,

$$\begin{aligned} & (|k|^2 - \lambda)(u, \varphi_k e_j^k)_{L(Q)} \\ & \quad = \quad (u, |k|^2 \varphi_k e_j^k)_{L(Q)} - (\lambda u, \varphi_k e_j^k)_{L(Q)} \qquad (4.14) \\ & \quad = \quad \Big((u, |k|^2 \varphi_k e_j^k)_{L(Q)} - a_Q(u, \varphi_k e_j^k)\Big) + \Big(a_Q(u, \varphi_k e_j^k) - (\lambda u, \varphi_k e_j^k)_{L(Q)}\Big) . \end{aligned}$$

This way of writing things will enable us, by means of Green's type formulae, to reduce our estimates to estimates in trace spaces, where the dimension of the underlying subsets of $\mathbb{R}^n$ being $n-1$ helps to get the desired powers of μ and λ for the order of the co-dimension of Z.

Before stating which Green's type formulae these are, we start to consider the two expressions within the larger brackets in (4.14) in a somewhat more general context:

- instead of the first one, we consider

$$(u, \mathcal{A}v)_{L(Q)} - a_Q(u, v) \tag{4.15}$$

  for all $u \in H(Q)$ and $v \in (H^2(Q))^n$, where $H^2(Q)$ stands for the Sobolev space of the functions of $L_2(Q)$ which have weak partial derivatives up to order two also in $L_2(Q)$, and $\mathcal{A}v = \mathcal{A}(v_j)_{j=1}^n \equiv (-\Delta v_j)_{j=1}^n$;

- instead of the second one, we consider

$$a_Q(u, v) - (\lambda u, v)_{L(Q)} \tag{4.16}$$

  for all $u \in Z_\lambda(Q)$ and $v \in V(Q) \equiv \{u \in H(Q) : \operatorname{div} u = 0\}$.

We deal first with (4.15).

Since the essential argument is already given in [10, pp.166-167], we may use Lemma 4.4 of that paper, namely that

$$\text{“}\forall u \in H^1(Q), \forall v \in H^2(Q), \tag{4.17}$$
$$\sum_{j=1}^n \left(\frac{\partial v}{\partial x_j}, \frac{\partial u}{\partial x_j} \right)_{L_2(Q)} - (-\Delta v, u)_{L_2(Q)} = (\tilde{\tau} v, \tilde{\gamma} u)_{\prod_{\ell=1}^{2n} L_2(F_\ell)} \text{”}.$$

Here, F_ℓ is the face of Q with equation $x_\ell = \pi$ if $\ell \in \{1, \dots, n\}$ and is the face of Q with equation $x_{\ell-n} = -\pi$ if $\ell \in \{n+1, \dots, 2n\}$; also,

$$\tilde{\gamma} : H^1(Q) \to \prod_{\ell=1}^{2n} H^{1/2}(F_\ell) \tag{4.18}$$

is the trace type operator given by $\tilde{\gamma}u = (u|_{F_\ell})_{\ell=1}^{2n}$, where $H^{1/2}(F_\ell)$ is a fractional Sobolev space, and

$$\tilde{\tau} : H^2(Q) \to \prod_{\ell=1}^{2n} H^{1/2}(F_\ell)$$

is the composition of $\tilde{\gamma}$ with derivation operators, as we define

$$\tilde{\tau}v = (D_\ell v|_{F_\ell})_{\ell=1}^{2n}\,,$$

with $D_\ell v = +\frac{\partial v}{\partial x_\ell}$ if $\ell \in \{1, \ldots, n\}$ and $D_\ell v = -\frac{\partial v}{\partial x_{\ell-n}}$ if $\ell \in \{n+1, \ldots, 2n\}$.

To simplify notation, we shall write $L_2(\partial Q)$ and $H^{1/2}(\partial Q)$ instead of $\prod_{\ell=1}^{2n} L_2(F_\ell)$ and $\prod_{\ell=1}^{2n} H^{1/2}(F_\ell)$, respectively. We shall also write, in line with the conventions set up in the beginning of this paper, $L(\partial Q)$ instead of $(L_2(\partial Q))^n$.

If we now consider $u = (u_i)_{i=1}^n \in H(Q)$ and $v = (v_i)_{i=1}^n \in (H^2(Q))^n$ and apply (4.17) above for each u_i and v_i, we get, on summing,

$$a_Q(v,u) - (\mathcal{A}v, u)_{L(Q)} = \sum_{i=1}^n (\tilde{\tau}v_i, \tilde{\gamma}u_i)_{L_2(\partial Q)},$$

or

$$a_Q(v,u) - (\mathcal{A}v, u)_{L(Q)} = (\tilde{\mathcal{T}}v, \tilde{\Gamma}u)_{L(\partial Q)}, \tag{4.19}$$

$\forall u \equiv (u_i)_{i=1}^n \in H(Q), v \equiv (v_i)_{i=1}^n \in (H^2(Q))^n$, if we define

$$\tilde{\mathcal{T}} : (H^2(Q))^n \to \left(H^{1/2}(\partial Q)\right)^n$$

by $\tilde{\mathcal{T}}v = (\tilde{\tau}v_i)_{i=1}^n$ and

$$\tilde{\Gamma} : H(Q) \to \left(H^{1/2}(\partial Q)\right)^n$$

by $\tilde{\Gamma}u = (\tilde{\gamma}u_i)_{i=1}^n$.

$\tilde{\mathcal{T}}$ and $\tilde{\Gamma}$ are, clearly, continuous linear operators.

We deal now with (4.16).

Consider the quotient space $T(Q) \equiv V(Q)/V_0(Q)$, let Γ be the natural projection of $V(Q)$ onto $T(Q)$, and let R be a lifting of Γ, i.e., a continuous linear operator from $T(Q)$ into $V(Q)$ such that $\Gamma \circ R = id_{T(Q)}$ (it is not difficult to see that such a R exists, taking account of the Hilbert structure of $V(Q)$).

Note that the kernel of $\tilde{\Gamma}|_{V(Q)}$ is $V_0(Q)$ (cf. [10, p.167]) and that this implies the injectivity of the operator $\tilde{\Gamma} \circ R$, allowing us to identify $T(Q)$ with a subspace of $\left(H^{1/2}(\partial Q)\right)^n$.

We have the following

Lemma **4.1** *There exists a continuous linear operator*

$$\mathcal{T}_\lambda : Z_\lambda(Q) \to T'(Q),$$

where the target space is the anti-dual of $T(Q)$, such that, for all $u \in Z_\lambda(Q)$, $v \in V(Q)$,

$$a_Q(u, v) - (\lambda u, v)_{L(Q)} = \langle \mathcal{T}_\lambda u, \Gamma v \rangle_{T'(Q) \times T(Q)} \tag{4.20}$$

Proof. Given $u \in Z_\lambda(Q)$, define $\mathcal{T}_\lambda u$ in the following way: for all $\varphi \in T(Q)$,

$$\langle \mathcal{T}_\lambda u, \varphi \rangle_{T'(Q) \times T(Q)} = a_Q(u, R\varphi) - (\lambda u, R\varphi)_{L(Q)}.$$

It is easy to see that this defines an operator $\mathcal{T}_\lambda$ with the desired properties.

We return now to (4.14). Using (4.19), (4.20), taking into account that $\mathcal{A}(\varphi_k e_j^k)$ $= |k|^2 \varphi_k e_j^k$, we can write

$$\begin{aligned} &(|k|^2 - \lambda)(u, \varphi_k e_j^k)_{L(Q)} \\ &\quad = \left((u, \mathcal{A}(\varphi_k e_j^k))_{L(Q)} - a_Q(u, \varphi_k e_j^k)\right) \\ &\qquad + \left(a_Q(u, \varphi_k e_j^k) - (\lambda u, \varphi_k e_j^k)_{L(Q)}\right) \\ &\quad = -\overline{(\tilde{\mathcal{T}}(\varphi_k e_j^k), \tilde{\Gamma} u)_{L(\partial Q)}} + \langle \mathcal{T}_\lambda u, \Gamma(\varphi_k e_j^k) \rangle_{T'(Q) \times T(Q)} \\ &\quad = \langle \mathcal{T}_\lambda u, \Gamma(\varphi_k e_j^k) \rangle_{T'(Q) \times T(Q)} - (\tilde{\Gamma} u, \tilde{\mathcal{T}}(\varphi_k e_j^k))_{L(\partial Q)}, \end{aligned} \tag{4.21}$$

for all $u \in Z_\lambda(Q)$.

Turning now to the consideration of the system $\{\varphi_k k\}_{k\in\mathbb{Z}^n}$, note first that we have

$$(u, \varphi_k k)_{L(Q)} = \hat{u}(k).k, \quad \forall u \in L(Q), \forall k \in \mathbb{Z}^n. \tag{4.22}$$

Note also that, for $u \equiv (u_j)_{j=1}^n \in H(Q)$,

$$\begin{aligned} \operatorname{div} u &= \sum_{j=1}^n \frac{\partial u_j}{\partial x_j} = \sum_{j=1}^n \sum_{k\in\mathbb{Z}^n} \widehat{\frac{\partial u_j}{\partial x_j}}(k)\varphi_k \\ &= \sum_{k\in\mathbb{Z}^n} \left(\sum_{j=1}^n \left(\frac{\partial u_j}{\partial x_j}, \varphi_k \right)_{L_2(Q)} \right) \varphi_k \\ &= \sum_{k\in\mathbb{Z}^n} \left(i\hat{u}(k).k + \sum_{\ell=1}^{2n} (\varepsilon_\ell u_\ell|_{F_\ell}, \varphi_k|_{F_\ell})_{L_2(F_\ell)} \right) \varphi_k \\ &= \sum_{k\in\mathbb{Z}^n} \left(i\hat{u}(k).k + (\sigma u, \tilde{\gamma}\varphi_k)_{L_2(\partial Q)} \right) \varphi_k \end{aligned} \tag{4.23}$$

in $L_2(Q)$, where

$$\varepsilon_\ell = \begin{cases} 1 & \text{if } \ell \in \{1, \dots, n\} \\ -1 & \text{if } \ell \in \{n+1, \dots, 2n\} \end{cases}, \tag{4.24}$$

$\tilde{\gamma}$ is the mapping in (4.18) and

$$\sigma : H(Q) \to H^{1/2}(\partial Q)$$

is defined as $\sigma u = (\varepsilon_\ell u_\ell|_{F_\ell})_{\ell=1}^{2n}$ (it is, obviously, a continuous linear operator). If $u \in V(Q)$ we have $\operatorname{div} u = 0$ and we can then say, in virtue of (4.23), that

$$\hat{u}(k).k = i(\sigma u, \tilde{\gamma}\varphi_k)_{L_2(\partial Q)}, \quad \forall k \in \mathbb{Z}^n,$$

or, taking (4.22) into account, that

$$(u, \varphi_k k)_{L(Q)} = i(\sigma u, \tilde{\gamma}\varphi_k)_{L_2(\partial Q)}, \quad \forall k \in \mathbb{Z}^n. \tag{4.25}$$

Next, define

W the space spanned by the $\tilde{\gamma}\varphi_k$ for $|k|^2 \leq \nu$;

X the space spanned by the $\Gamma(\varphi_k e_j^k)$ for $|k|^2 \leq \nu$;

Y the space spanned by the $\tilde{\mathcal{T}}(\varphi_k e_j^k)$ for $|k|^2 \leq \nu$;

G the space generated by the $\varphi_k e_j^k$ for $|k|^2 = \lambda$.

It follows immediately from (4.21), (4.25), the definition of Z and the fact that, for each $k \in \mathbb{Z}^n \setminus \{0\}$, the vectors e_j^k constitute, together with k, a basis of $\mathbb{C}^n$, that the space

$$\begin{aligned} Z_0 \;\equiv\; \{&u \in Z_\lambda(Q) : u \in G^\perp \,\text{in}\, L(Q),\, \tilde{\Gamma}u \in Y^\perp \,\text{in}\, L(\partial Q), \\ &\mathcal{T}_\lambda u \in X^\circ \,\text{in}\, T'(Q),\, \sigma u \in W^\perp \,\text{in}\, L_2(\partial Q)\} \end{aligned}$$

is a closed subspace of Z. This implies that

$$\operatorname{codim}_{Z_\lambda(Q)} Z \leq \operatorname{codim}_{Z_\lambda(Q)} Z_0 \leq \dim G + \dim Y + \dim X + \dim W, \tag{4.26}$$

so that we are going to prove our initial claim on the first co-dimension by means of a suitable control of the others.

This control reduces, however, to a counting problem similar to one Métivier faced in the elliptic setting — cf. [10, pp. 169-170]. We briefly sketch how it can be done.

In the case of $\dim G$ we clearly have

$$\dim G \leq c_n(1 + \lambda^{(n-1)/2}), \tag{4.27}$$

where c_n is a positive number depending only upon n.

In the case of $\dim X$, define $\tilde{X}$ to be the space spanned by $\tilde{\Gamma}(\varphi_k e_j^k)$ for $|k|^2 \leq \nu$, and note that $\dim \tilde{X} = \dim X$: in fact, the maximum number of linearly independent elements is the same in both spaces, as we can easily see by using the observation $\ker \tilde{\Gamma}|_{V(Q)} = V_0(Q)$ (made just before Lemma 4.1).

We now estimate $\dim \tilde{X}$. Introduce the notations

$$e_j^k \equiv (e_{jm}^k)_{m=1}^n,$$

$$\widehat{k_\ell} \equiv (k_1, \ldots, k_{\ell-1}, k_{\ell+1}, \ldots, k_n) \quad \text{if} \quad \ell \in \{1, \ldots, n\},$$
$$\widehat{k_\ell} \equiv (k_1, \ldots, k_{\ell-n-1}, k_{\ell-n+1}, \ldots, k_n) \quad \text{if} \quad \ell \in \{n+1, \ldots, 2n\},$$

and analogously for $\widehat{x_\ell}$,

$$k_\ell \equiv k_{\ell-n} \quad \text{if} \ \ \ell \in \{n+1, \ldots, 2n\}.$$

Note that we can write, using these notations and the one established in (4.24),

$$\tilde{\Gamma}(\varphi_k e_j^k) = (2\pi)^{-n/2} \sum_{\ell=1}^{2n} \sum_{m=1}^{n} e_{jm}^k \exp(ik_\ell \varepsilon_\ell \pi) \left((\exp(i\widehat{k_\ell}.\widehat{x_\ell})\delta_{\ell r})_{r=1}^{2n} \delta_{ms} \right)_{s=1}^{n},$$

so that each $\tilde{\Gamma}(\varphi_k e_j^k)$ in $\tilde{X}$ is a linear combination of elements of $\left(H^{1/2}(\partial Q)\right)^n$ of the form

$$\left((\exp(i\widehat{k_\ell}.\widehat{x_\ell})\delta_{\ell r})_{r=1}^{2n} \delta_{ms} \right)_{s=1}^{n}, \tag{4.28}$$

where $m \in \{1, \ldots, n\}$, $\ell \in \{1, \ldots, 2n\}$ and $\widehat{k_\ell}$ must satisfy the relation $|\widehat{k_\ell}|^2 \leq \nu$. We can then estimate $\dim \tilde{X}$ by counting the number of such elements. Clearly, the number of different possibilities for the $\widehat{k_\ell}$ cannot exceed $(2\nu^{1/2}+1)^{n-1}$ and the number of different possibilities for $((\delta_{\ell r})_{r=1}^{2n}\delta_{ms})_{s=1}^n$ is $2n^2$. Therefore, the number of elements of the form (4.28) does not exceed

$$c_n(1 + \nu^{(n-1)/2}), \tag{4.29}$$

where c_n depends only on n.

In the case of $\dim Y$, we proceed in a way similar to the estimation of $\dim \tilde{X}$. Using the same notations, we can write

$$\tilde{\mathcal{T}}(\varphi_k e_j^k) = (2\pi)^{-n/2} \sum_{\ell=1}^{2n} \sum_{m=1}^{n} -e_{jm}^k \exp(ik_\ell \varepsilon_\ell \pi) \varepsilon_\ell i k_\ell \left((\exp(i\widehat{k_\ell}.\widehat{x_\ell})\delta_{\ell r})_{r=1}^{2n} \delta_{ms} \right)_{s=1}^{n},$$

so that we get the same upper estimate

$$c_n(1 + \nu^{(n-1)/2}) \tag{4.30}$$

for $\dim Y$.

Finally, in the case of $\dim W$ we can also benefit from what was done in estimating $\dim \tilde{X}$. Using the same notations, we can write

$$\tilde{\gamma}\varphi_k = (2\pi)^{-n/2} \sum_{\ell=1}^{2n} \exp(ik_\ell \varepsilon_\ell \pi) \left(\exp(i\widehat{k_\ell}.\widehat{x_\ell})\delta_{\ell r} \right)_{r=1}^{2n},$$

so that W can be generated by $\{\exp(i\widehat{k_\ell}.\widehat{x_\ell})(\delta_{\ell r})_{r=1}^{2n}\}_{\substack{\ell=1,\ldots,2n \\ |\widehat{k_\ell}| \le \nu}}$. We can then say that

$$c_n(1 + \nu^{(n-1)/2}) \tag{4.31}$$

is an upper estimate for $\dim W$, with c_n depending only on n.

Putting (4.27), (4.29), (4.30) and (4.31) into (4.26) we get

$$\operatorname{codim}_{Z_\lambda(Q)} Z \le c_n(1 + \lambda^{(n-1)/2} + \nu^{(n-1)/2}) \tag{4.32}$$

(c_n has been redefined, of course, but still depends only on n).

We shall delay the choice of ν till later on.

We recall that the objective of this subsection is to find a closed subspace E of $Z_\lambda(Q)$ satisfying (4.13) and with a suitable bound on $\operatorname{codim}_{Z_\lambda(Q)}(E)$. We are going to show that such an E can be taken as

$$E \equiv \{u \in Z : \tilde{\Gamma} u \in \mathcal{E}\} \tag{4.33}$$

where $\mathcal{E}$ is a closed subspace of $\left(H^{1/2}(\partial Q)\right)^n$ such that, for all $u \in \mathcal{E}$ and all $v \in \left(H^{1/2}(\partial Q)\right)^n$,

$$|(u,v)_{L(\partial Q)}| \leq \nu^{-1/2} \|u\|_{\left(H^{1/2}(\partial Q)\right)^n} \|v\|_{\left(H^{1/2}(\partial Q)\right)^n} \tag{4.34}$$

and $\operatorname{codim} \mathcal{E} \leq K\nu^{(n-1)/2}$, where $K(>0)$ does not depend on the $\nu(\geq 0)$ considered.

We must, of course, first of all ensure that there exists such an $\mathcal{E}$. The essential step is already done in [10, p.168], namely that there is a constant $k(>0)$ such that for all $\nu \geq 0$ there exists a closed subspace $\mathcal{H}$ of $H^{1/2}(\partial Q)$ of co-dimension bounded above by

$$k\nu^{(n-1)/2}$$

and such that, for all $u \in \mathcal{H}$ and all $v \in H^{1/2}(\partial Q)$,

$$|(u,v)_{L_2(\partial Q)}| \leq \nu^{-1/2} \|u\|_{H^{1/2}(\partial Q)} \|v\|_{H^{1/2}(\partial Q)}$$

(we would like to remark that the proof of this result relies on the estimates of [6] for the Kolmogorov diameters of embeddings of fractional Sobolev spaces into L_p-spaces). It is straightforward to show that if we define $\mathcal{E} \equiv \mathcal{H}^n$ we get a space $\mathcal{E}$ with the properties required above (in particular, K can be taken equal to nk).

Returning then to the space E defined in (4.33), it is clear that E is a closed subspace of $Z_\lambda(Q)$, the co-dimension of which is

$$\begin{aligned} \operatorname{codim}_{Z_\lambda(Q)} E &\leq \dim \frac{Z_\lambda(Q)}{Z} + \dim \frac{Z_\lambda(Q)}{Z_\lambda(Q) \cap \tilde{\Gamma}^{-1}(\mathcal{E})} \\ &\leq c_n(1 + \lambda^{(n-1)/2} + \nu^{(n-1)/2}) + K\nu^{(n-1)/2}, \end{aligned}$$

in view of (4.32) and the choice of $\mathcal{E}$. That is,

$$\operatorname{codim}_{Z_\lambda(Q)} E \leq C(1 + \lambda^{(n-1)/2} + \nu^{(n-1)/2}), \tag{4.35}$$

where C is independent of λ and ν.

From this we see that we get the required result for the co-dimension of E (cf. phrase after (4.13)) if we choose ν proportional to μ (which is what we are going to do in the end).

Our next task is therefore to prove that (4.13) holds for a suitable choice of ν proportional to μ, which can be done by a reasoning entirely similar to the one used in the elliptic setting — cf. [10, pp. 171-172]. We refer only the main points.

Given $u \in E$, define

$$v \equiv \sum_{k,j} \frac{(u, \varphi_k e_j)_{L(Q)}}{|k|^2 - \lambda} \varphi_k e_j \qquad \text{in} \quad L(Q), \tag{4.36}$$

where the summation runs over all $k \in \mathbb{Z}^n$ such that $|k|^2 > \nu$ and all $j \in \{1, \ldots, n\}$. Actually, in order that this be well-defined, we shall require that $\nu \geq 2\lambda \geq 2$. Moreover, it is then a standard exercise in Hilbert space techniques to show that $v \in (H^2(Q))^n$ and that, indeed, the series in (4.36) converges to v in this latter space. As a consequence we have

$$(\mathcal{A} - \lambda)v = u \qquad \text{in} \quad L(Q) \quad \text{for} \quad u \in E. \tag{4.37}$$

Note also that, for such u and v,

$$(u, v)_{L(Q)} = \sum_{k,j} \frac{|(u, \varphi_k e_j)_{L(Q)}|^2}{|k|^2 - \lambda} \geq 0, \tag{4.38}$$

so that, with the help of (4.19) and (4.34), we can write

$$\|u\|^2_{L(Q)} \leq c\nu^{-1/2} \|u\|_{H(Q)} \|v\|_{(H^2(Q))^n} + \|u\|_{H(Q)} \|v\|_{H(Q)}, \tag{4.39}$$

where c is a constant that may be taken as the product of the norms of $\tilde{\Gamma}$ and $\tilde{\mathcal{T}}$. Since $\|v\|^2_{H(Q)} \leq 6\nu^{-1} \|u\|^2_{L(Q)}$ and $\|v\|^2_{(H^2(Q))^n} \leq 7\|u\|^2_{L(Q)}$, we get

$$\|u\|^2_{H(Q)} \geq c\nu \|u\|^2_{L(Q)},$$

where, of course, the constant $c(<1)$ has been redefined.

Choosing now $\nu = \frac{2\mu}{c}$ (assuming that $\mu \geq c\lambda$) we finally get

$$(u,u)_{H(Q)} - \mu(u,u)_{L(Q)} \geq \varepsilon \|u\|^2_{L(Q)} \tag{4.40}$$

for all $u \in E$, as required in (4.13) (we can choose $\varepsilon = \mu$ in (4.40)), so that, recalling also (4.35),

$$N(\mu, Z_\lambda(Q), L(Q), (\cdot,\cdot)_{H(Q)}) \leq c'(1 + \lambda^{(n-1)/2} + \mu^{(n-1)/2}) \tag{4.41}$$

for $\mu \geq c\lambda \geq c$. Actually, due to the monotonicity of the counting function with respect to μ, (4.41) holds for all $\mu \geq 0$ and $\lambda \geq 1$, possibly by redefinition of c', and from it we get, using the translation and scaling argument in (4.6),

$$N(\mu, Z_\lambda(J), L(J), (\cdot,\cdot)_{H(J)}) \leq c'' \left(1 + (\delta/(2\pi))^{n-1}(\lambda^{(n-1)/2} + \mu^{(n-1)/2})\right) \tag{4.42}$$

for $0 < \delta \leq 2\pi$ and $\mu, \lambda \geq (2\pi)^2\delta^{-2}$.

4.5 The result when Ω is a cube

Denoting, as in subsection 4.1, by J an arbitrary n-cube in $\mathbb{R}^n$ with side length δ, we can write, due to (4.1) and (4.12),

$$N(\lambda, V_0(J), L(J), a_J) - \frac{|J|_n}{(2\pi)^n}(n-1)|B^n|_n\lambda^{n/2} \leq c_n(1 + (\delta/(2\pi))^{n-1}\lambda^{(n-1)/2})$$

($\lambda \geq 0$) and, due to (4.2), (4.12), (4.4) and (4.42),

$$N(\lambda, V_0(J), L(J), a_J) - \frac{|J|_n}{(2\pi)^n}(n-1)|B^n|_n\lambda^{n/2} \geq -C\left(1 + (\delta/(2\pi))^{n-1}\lambda^{(n-1)/2}\right)$$

($0 < \delta \leq 2\pi\,;\ \lambda \geq (2\pi)^2\delta^{-2}$). That is, there exists some positive constant c such that

$$\left| N(\lambda, V_0(J), L(J), a_J) - \frac{|J|_n}{(2\pi)^n}(n-1)|B^n|_n\lambda^{n/2} \right| \leq c\,\delta^{n-1}\lambda^{(n-1)/2} \tag{4.43}$$

for all $\delta \in (0, 2\pi]$ and all $\lambda \geq (2\pi)^2\delta^{-2}$.

5 Estimates near the boundary

As promised at the end of section 3, we are now going to estimate

$$N(\mu, \mathcal{V}_0(\omega_r), L(\omega_r), (\cdot,\cdot)_{\mathcal{V}_0(\omega_r)}) \tag{5.1}$$

from above.

In what follows, r will stand for an arbitrary non-negative integer greater than or equal to r_0 (cf.beginning of section 3).

Note that

$$\mathcal{V}_0(\omega_r) \equiv \{u|_{\omega_r} : u \in V_0(\Omega)\} \subset (\mathcal{H}_0^1(\omega_r))^n \equiv \mathcal{H}_0(\omega_r), \tag{5.2}$$

where $\mathcal{H}_0^1(\omega_r)$ denotes the set of restrictions to ω_r of the elements of $H_0^1(\Omega)$. We know from [10, p.146] (cf. also subsection 2.1) that $\mathcal{H}_0^1(\omega_r)$ is a Hilbert space the norm of which is given by

$$\|v\|_{\mathcal{H}_0^1(\omega_r)} \equiv \inf_{\substack{u\in H_0^1(\Omega)\\ u|_{\omega_r}=v}} \|u\|_{H_0^1(\Omega)}, \qquad \forall v \in \mathcal{H}_0^1(\omega_r), \tag{5.3}$$

and that

$$(\mathcal{H}_0^1(\omega_r), L_2(\omega_r), (\cdot,\cdot)_{\mathcal{H}_0^1(\omega_r)})$$

is a variational triplet. As a consequence we can state, with the help of [10, Prop. 2.8], that

$$(\mathcal{H}_0(\omega_r), L(\omega_r), (\cdot,\cdot)_{\mathcal{H}_0(\omega_r)})$$

is a variational triplet (where $(\cdot,\cdot)_{\mathcal{H}_0(\omega_r)} \equiv \sum_{j=1}^n (\cdot_j, \cdot_j)_{\mathcal{H}_0^1(\omega_r)}$, as expected) and, for all $\mu \in \mathbb{R}$,

$$N(\mu, \mathcal{H}_0(\omega_r), L(\omega_r), (\cdot,\cdot)_{\mathcal{H}_0(\omega_r)}) = nN(\mu, \mathcal{H}_0^1(\omega_r), L_2(\omega_r), (\cdot,\cdot)_{\mathcal{H}_0^1(\omega_r)}). \tag{5.4}$$

We would like now to compare the counting function in (5.1) with the counting function on the left-hand side of (5.4). In order to do that, we use the characterization given in [10, Prop. 2.2], which allows us to write, for $\mu > 0$,

$$N(\mu, \mathcal{V}_0(\omega_r), L(\omega_r), (\cdot,\cdot)_{\mathcal{V}_0(\omega_r)}) = \#\{k \in \mathbb{N}_0 : d_k(B_{\mathcal{V}_0(\omega_r)}, L(\omega_r)) \geq \mu^{-1/2}\}$$

and

$$N(\mu, \mathcal{H}_0(\omega_r), L(\omega_r), (\cdot,\cdot)_{\mathcal{H}_0(\omega_r)}) = \#\{k \in \mathbb{N}_0 : d_k(B_{\mathcal{H}_0(\omega_r)}, L(\omega_r)) \geq \mu^{-1/2}\},$$

where the B's denote the closed unit balls of the corresponding spaces and d_k stands for the k-th diameter of Kolmogorov (for a precise definition, refer to [10, p. 130]). In this way, the comparison follows from a corresponding comparison between these balls: since, as it is easily seen, $B_{\mathcal{V}_0(\omega_r)} \subset B_{\mathcal{H}_0(\omega_r)}$, then $d_k(B_{\mathcal{V}_0(\omega_r)}, L(\omega_r)) \leq d_k(B_{\mathcal{H}_0(\omega_r)}, L(\omega_r))$, $\forall k \in \mathbb{N}_0$, and finally

$$N(\mu, \mathcal{V}_0(\omega_r), L(\omega_r), (\cdot,\cdot)_{\mathcal{V}_0(\omega_r)} \leq N(\mu, \mathcal{H}_0(\omega_r), L(\omega_r), (\cdot,\cdot)_{\mathcal{H}_0(\omega_r)}), \tag{5.5}$$

for all $\mu > 0$.

Observe that this argument avoids explicit consideration of the relation between the factor spaces given by the divergence being zero.

In view of (5.5) and (5.4), it would now be convenient to have an estimate for $N(\mu, \mathcal{H}_0^1(\omega_r), L_2(\omega_r), (\cdot,\cdot)_{\mathcal{H}_0^1(\omega_r)})$. It is, in fact, already known (see [10, p.151])[3] that, for all $\mu > 0$,

$$N(\mu, \mathcal{H}_0^1(\omega_r), L_2(\omega_r), (\cdot,\cdot)_{\mathcal{H}_0^1(\omega_r)}) \leq c_n |[\omega_r]_{\mu^{-1/2}}|_n \mu^{n/2}, \tag{5.6}$$

where c_n depends only on n and

$$[\omega_r]_{\mu^{-1/2}} \equiv \{x \in \Omega : \mathrm{dist}\,(x, \omega_r) < \sqrt{n}\mu^{-1/2}\}\,.$$

[3]The counting function in (5.6) is not exactly the same as the counting function featured in the paper cited, but an argument similar to the one used to get (5.5) takes one through to (5.6).

In order to connect this estimate with our hypothesis in Theorem 1.2, recall that

$$\omega_r \subset (\partial\Omega)_{(\sqrt{n}+1)2^{-r}}$$

(cf. (3.2)) and, consequently,

$$[\omega_r]_{\mu^{-1/2}} \subset (\partial\Omega)_{(\sqrt{n}+1)2^{-r}+\sqrt{n}\mu^{-1/2}}. \tag{5.7}$$

Using this in (5.6) and putting the result in (5.4) and then in (5.5) we obtain, possibly by redefining c_n,

$$N(\mu, \mathcal{V}_0(\omega_r), L(\omega_r), (\cdot,\cdot)_{\mathcal{V}_0(\omega_r)} \leq c_n |(\partial\Omega)_{(\sqrt{n}+1)2^{-r}+\sqrt{n}\mu^{-1/2}}|_n \mu^{n/2} \tag{5.8}$$

for all $\mu > 0$ and all integers $r \geq r_0$.

In the particular case $\mu = 2(\lambda + 1)$, which is needed in (3.5), it is easy to get from (5.8) the estimate

$$N(2(\lambda+1), \mathcal{V}_0(\omega_r), L(\omega_r), (\cdot,\cdot)_{\mathcal{V}_0(\omega_r)}) \leq c'_n |(\partial\Omega)_{(\sqrt{n}+1)2^{-r}+\sqrt{n}\lambda^{-1/2}}|_n \lambda^{n/2} \tag{5.9}$$

for all $\lambda \geq 1$ and all integers $r \geq r_0$.

6 Assembling things together

We are now able to complete the line of thought initiated in section 3.

Recall that Ω is an arbitrary bounded open non-empty subset of $\mathbb{R}^n$. Recall also the decomposition of Ω that we made in the beginning of section 3 by means of tessellations of $\mathbb{R}^n$. The way we take advantage of those tessellations, of the decomposition of the problem made in section 3 and of the estimates obtained in sections 4 and 5 is the same as in the case of the Dirichlet Laplacian — cf. [10] or [8].

Using (3.1) and (4.43) we can write, for all integers $r \geq r_0$ and all $\lambda \geq (2\pi)^2 2^{2r}$,

$$\begin{aligned}
&N(\lambda, V_0(\Omega), L(\Omega), a) - \frac{|\Omega|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2} \\
\geq\ & \sum_{\rho,\nu} c\, 2^{-\rho(n-1)} \lambda^{(n-1)/2} - \frac{n-1}{(2\pi)^n}|B^n|_n |(\partial\Omega)_{(\sqrt{n}+1)2^{-r}}|_n \lambda^{n/2} \\
=\ & -c\left(\sum_{\rho=r_0}^{r} 2^{-\rho(n-1)}(\# A_\rho)\right)\lambda^{(n-1)/2} - \frac{n-1}{(2\pi)^n}|B^n|_n |(\partial\Omega)_{(\sqrt{n}+1)2^{-r}}|_n \lambda^{n/2} \qquad (6.1) \\
\geq\ & -c\left(\sum_{\rho=r_0}^{r} 2^{\rho}|(\partial\Omega)_{(\sqrt{n}+1)2^{-\rho+1}}|_n\right)\lambda^{(n-1)/2} - \frac{n-1}{(2\pi)^n}|B^n|_n |(\partial\Omega)_{(\sqrt{n}+1)2^{-r}}|_n \lambda^{n/2},
\end{aligned}$$

where this latter inequality follows from

$$(\# A_\rho).2^{-\rho n} = \begin{cases} |\Omega_\rho \setminus \Omega_{\rho-1}|_n & \text{if } \rho > r_0 \\ |\Omega_\rho|_n & \text{if } \rho = r_0 \end{cases} \leq \begin{cases} |\omega_{\rho-1}|_n & \text{if } \rho > r_0 \\ |\Omega|_n & \text{if } \rho = r_0 \end{cases} \qquad (6.2)$$

together with the inclusion (3.2) (the case $\rho = r_0$ can be included in the main stream by adjusting the constant c).

By this same token, and using also (3.3), (4.43), (3.5), (5.9) and (4.42), we can write, for all integers $r \geq r_0$ and all $\lambda \geq (2\pi)^2 2^{2r}$,

$$\begin{aligned}
&N(\lambda, V_0(\Omega), L(\Omega), a) - \frac{|\Omega|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2} \\
\leq\ & c\left(\sum_{\rho=r_0}^{r} 2^{\rho}|(\partial\Omega)_{(\sqrt{n}+1)2^{-\rho+1}}|_n\right)\lambda^{(n-1)/2} + c'_n |(\partial\Omega)_{(\sqrt{n}+1)2^{-r}+\sqrt{n}\lambda^{-1/2}}|_n \lambda^{n/2} \\
& + \sum_{\rho,\nu} c'' \left(1 + \frac{2^{-\rho(n-1)}}{(2\pi)^{n-1}}(\lambda^{(n-1)/2} + 2^{(n-1)/2}(\lambda+1)^{(n-1)/2})\right) \qquad (6.3) \\
\leq\ & c'\left(\sum_{\rho=r_0}^{r} 2^{\rho}|(\partial\Omega)_{(\sqrt{n}+1)2^{-\rho+1}}|_n\right)\lambda^{(n-1)/2} + c'_n |(\partial\Omega)_{(\sqrt{n}+1)2^{-r}+\sqrt{n}\lambda^{-1/2}}|_n \lambda^{n/2}.
\end{aligned}$$

From this and (6.1) we get, for some constant c and for all integers $r \geq r_0$ and all $\lambda \geq (2\pi)^2 2^{2r}$,

$$\begin{aligned}
&\left| N(\lambda, V_0(\Omega), L(\Omega), a) - \frac{|\Omega|_n}{(2\pi)^n}(n-1)|B^n|_n \lambda^{n/2} \right| \qquad (6.4) \\
\leq\ & c\left[\left(\sum_{\rho=0}^{r} 2^{\rho}|(\partial\Omega)_{(\sqrt{n}+1)2^{-\rho+1}}|_n\right)\lambda^{(n-1)/2} + |(\partial\Omega)_{(\sqrt{n}+1)2^{-r}+\sqrt{n}\lambda^{-1/2}}|_n \lambda^{n/2}\right].
\end{aligned}$$

It follows now immediately that, if $|\partial\Omega|_n = 0$,

$$\lim_{\lambda\to+\infty} \frac{N(\lambda, V_0(\Omega), L(\Omega), a)}{\lambda^{n/2}} = \frac{|\Omega|_n}{(2\pi)^n}(n-1)|B^n|_n\,, \tag{6.5}$$

a result that, as far as we know, was only previously obtained in the case when the boundary of Ω is smooth. Thus, our approach has shown that (6.5) holds true for all bounded open non-empty subsets Ω of $\mathbb{R}^n$ the boundary of which have n-dimensional Lebesgue measure equal to 0.

We may now conclude the proof of Theorem 1.2. Indeed, the deduction of the result from the estimate (6.4) is similar to the case of the Dirichlet Laplacian, so that we refer the reader to [3, proof of the Prop. of section 5] for the case $d > n-1$ and to [10, pp. 198-199] for the case $d = n-1$ (for the special case of Theorem 1.2 covered by Corollary 1.3 the reader may also wish to compare with [8, pp. 499-507]).

Acknowledgement: The author would like to thank the referee for the improvement of the presentation.

References

[1] K. I. Babenko, "On the asymptotics of the eigenvalues of the linearized Navier-Stokes equations", *Dokl. Acad. Nauk SSSR* 263 (1982) 521-524; English transl. in *Soviet Math. Dokl.* 25 (1982).

[2] A. M. Caetano, "Asymptotic Distribution of Weyl Numbers and Eigenvalues," D.Phil. thesis, University of Sussex 1991.

[3] A. M. Caetano, "On the search for the asymptotic behaviour of the eigenvalues of the Dirichlet Laplacian for bounded irregular domains", *International Journal of Applied Science & Computations* 2 (1995) 261-287.

[4] P. Constantin, C. Foias, *Navier-Stokes Equations* (The University of Chicago Press, Chicago, 1988).

[5] D. E. Edmunds, W. D. Evans, *Spectral theory and differential operators* (Clarendon Press, Oxford, 1987).

[6] A. El Kolli, "n-ième épaisseur dans les espaces de Sobolev", *J. Approx. Theory* 10 (1974).

[7] A. N. Kozhevnikov, "On the operator of the linearized steady-state Navier-Stokes problem", *Math. Sbornik* 125(167) (1984); English transl. in *Math. USSR Sb.* 53 (1986) 1-16.

[8] M. Lapidus, "Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl-Berry conjecture", *Trans. Amer. Math. Soc.* 325 (1991) 465-529.

[9] S. Z. Levendorskiĭ, "Asymptotics of the spectrum of problems with constraints", *Math. Sbornik* 129(171) (1986); English transl. in *Math. USSR Sb.* 57 (1987) 77-95.

[10] G. Métivier, "Valeurs propres de problèmes aux limites elliptiques irréguliers", *Bull. Soc. Math. France*, Mémoire 51-52 (1977) 125-219.

[11] G. Métivier, "Valeurs propres d'opérateurs définis par la réstriction de systèmes variationnels à des sous-espaces", *J. Math. Pures Appl.* 57 (1978) 133-156.

[12] R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis* (North-Holland, Amsterdam, 1985).

Departamento de Matemática
Universidade de Aveiro
3810 AVEIRO
Portugal
e-mail: acaetano@mat.ua.pt