GROWTH ENVELOPES IN FUNCTION SPACES

Lecture of synthesis

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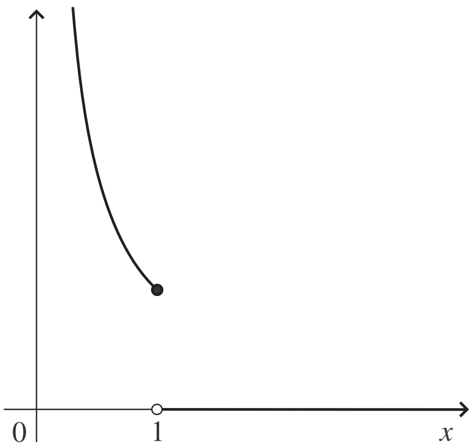
1 Introduction

The problem of determining the local growth envelope function of a function space containing unbounded functions consists, roughly speaking, in obtaining the fastest local growth which the functions from that space can tolerate. In other words, it is the problem of studying the worst possible singularity that a function from some given function space can have.

We give an example:



In the figure we see represented the function

$$x^{-1/p}\chi_{]0,1]},$$

which *does not belong* to $L_p(\mathbb{R})$, $0 < p < \infty$. Nevertheless, no matter how small we take the $\varepsilon > 0$, the function

$$x^{-(1-\varepsilon)/p}\chi_{]0,1]}$$

already *belongs* to $L_p(\mathbb{R})$. Therefore, the function $x^{-1/p}\chi_{]0,1]}$ has some of the properties of what we would like to call a local growth envelope function of the space $L_p(\mathbb{R})$. It has, however, the disadvantage of describing only what happens with respect to a particular point and of having itself a particular form coming from the special type of functions considered in this example.

The next step will then be to find a way of measuring the “maximum” ability of local growth without falling in this type of particularizations.

An adequate tool is the so-called decreasing rearrangement of a function: given any measurable function $f : \mathbb{R}^n \to \mathbb{C}$, its decreasing rearrangement is defined as

$$f^*(t) := \inf\{\lambda \geq 0 : m_f(\lambda) \leq t\}, \quad t \geq 0,$$

where

$$m_f(\lambda) := |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|, \quad \lambda \geq 0.$$

Notice that

- $f^*(t) \in [0, \infty]$;
- f^* is decreasing;
- f^* is continuous from the right;

- f and f^* are equimeasurable, that is, for each $\lambda \geq 0$, $m_f(\lambda)$ coincides with $|\{t \in \mathbb{R}_0^+ : |f^*(t)| > \lambda\}|$.

Thus, in natural language we can say that f^* rearranges f in such a way that the greater values come first, that is, near zero. Consequently, the behaviour of f^* near zero give us an indication about the ability of local growth for f.

Returning to the example of $L_p(\mathbb{R})$, observing the behaviour of the function

$$t \mapsto \sup_{\|f|L_p(\mathbb{R})\| \leq 1} f^*(t), \quad t > 0, \tag{1}$$

near zero we get an indication about the "maximum" ability of local growth for the functions of $L_p(\mathbb{R})$ (notice that if we hadn't normalized the functions f to be considered in (1), the definition would not make sense!). In fact, the following promising result holds:

Proposition 1.1 *Given* $0 < p < \infty$, $\quad \sup_{\|f|L_p(\mathbb{R})\| \leq 1} f^*(t) = t^{-1/p}, \quad t > 0.$

Later on we will sketch a proof for this result. For the moment we would like to introduce the following general definition justified by the considerations done until now:

Definition 1.2 *The growth envelope function of a quasi-normed space A of (classes of equivalence of) locally integrable complex functions in* $\mathbb{R}^n$ *is a function defined by*

$$\mathcal{E}_{LG}|A(t) := \sup_{\|f|A\| \leq 1} f^*(t), \quad \textit{for small } t > 0$$

or the class of equivalence of such functions.

The concept was introduced by Haroske [17] (cf. also [18] and [25]) in 2001, under the influence of works of Edmunds and Triebel [13] and Triebel [24] and having as forerunners works of Netrusov [22], [23]. As Haroske herself refers, it is related with the concept of fundamental function, from the theory of rearrangement invariant spaces. As we will see, it is related with embeddings of Sobolev type and refinements obtained along several years by various authors, specially if we complement the notion of growth envelope function with the notion of fine index (it is to both things together that we call growth envelope), about which we will briefly talk in the last part of this lecture.

Besides its intrinsic interest, as it allows the classification of function spaces according to the ability of local growth of its elements, the concept of growth envelope has been successfully used in the determination of necessary conditions for the existence of embeddings between function spaces. There exists also a surprising connection with the concept of approximation numbers which allows in some cases to obtain sharp upper estimates for the latter.

2 Historically-oriented survey of results

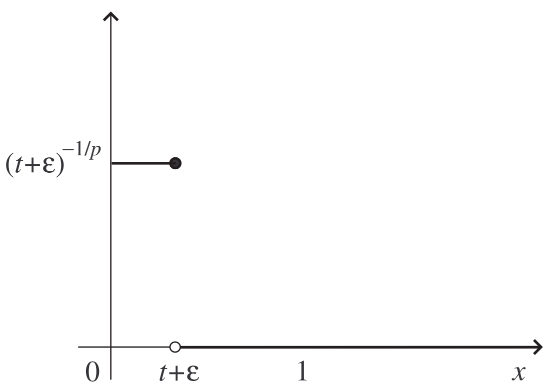
We are here interested in the growth envelope functions of function spaces of Besov and Triebel-Lizorkin type. General properties of growth envelopes were established by Haroske in [17], but we will only refer to them when there is a need for that. In that same paper, Haroske also obtained the growth envelopes of more elementary function spaces. We will not describe here those results (with the exception of the case of L_p-spaces), though they were used in some of the proofs of results we would like to present (the more general results, however, are obtained through techniques that do not use such preliminary results, this justifying the omission of the latter here).

It is not our intention to refer here the various authors who have contributed for the construction of the various results to be presented. Such information can be seen in [25, Cap. II] and in the list of references included in that book. Our idea here is to start telling the story since the time when the concept of growth envelope was introduced, but we should mention that its invention was directly influenced by the works [13] of Edmunds and Triebel and [24] of Triebel. These have also influenced decisively the results we would like to take as starting point, namely the determination of the growth envelope functions for Besov spaces $B^s_{pq}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $F^s_{pq}(\mathbb{R}^n)$. We will have, however, opportunity to refer immediately after to some particular cases where it is apparent the influence of results well-known to the mathematical analyst in the study of the problem in question.

In order one can appreciate what is at stake to be proved, we start by sketching the proof of the preliminary result contained in Proposition 1.1:

$$1 \geq \|f|L_p(\mathbb{R})\| = \left(\int_0^\infty f^*(s)^p\,ds\right)^{1/p} \geq \left(\int_0^t f^*(s)^p\,ds\right)^{1/p} \geq (t f^*(t)^p)^{1/p},$$

and this guarantees that $\sup_{\|f|L_p(\mathbb{R})\|\leq 1} f^*(t) \leq t^{-1/p}, \quad t>0.$



For the opposite inequality, we take, for each $t \in]0,1[$ and $\varepsilon > 0$,

$$f_{t,\varepsilon}(x) = (t+\varepsilon)^{-1/p}\chi_{]0,t+\varepsilon]},$$

so that we have

$$f^*_{t,\varepsilon}(s) = (t+\varepsilon)^{-1/p}\chi_{[0,t+\varepsilon[}$$

and the required conclusion, after letting ε go to zero.

As hereafter all spaces will be considered over $\mathbb{R}^n$, we shall omit this information in the notation of the spaces. On the other hand, the general restrictions for the parameters s, p, q (that is, the ones which hold by default) are $s \in \mathbb{R}$ and $0 < p, q \leq \infty$ (with the further restriction $p < \infty$ in the case of spaces of Triebel-Lizorkin type).

We consider now the, say, classical Besov and Triebel-Lizorkin spaces. For these, Haroske [17] and Triebel [25] proved in 2001 the following:

Proposition 2.1 *Assuming that $s > n(\frac{1}{p} - 1)_+$,*

1. *if $s < \frac{n}{p}$ (the subcritical case),*

$$\mathcal{E}_{LG}|B^s_{p,q}(t), \mathcal{E}_{LG}|F^s_{p,q}(t) \sim t^{\frac{s}{n} - \frac{1}{p}} \quad \textit{near } 0;$$

2. *if $s = \frac{n}{p}$ (the critical case),*

$$\mathcal{E}_{LG}|B^{\frac{n}{p}}_{p,q}(t) \sim |\log t|^{\frac{1}{q'}} \quad \textit{near 0}, \ \textit{assuming further that } q > 1;$$

$$\mathcal{E}_{LG}|F^{\frac{n}{p}}_{p,q}(t) \sim |\log t|^{\frac{1}{p'}} \quad \textit{near 0}, \ \textit{assuming further that } p > 1;$$

Here and hereafter, any expression like

$$f(t) \sim g(t) \ \text{ near } 0$$

means that there exists $\varepsilon > 0$ and constants $c_1, c_2 > 0$ such that

$$c_1 g(t) \leq f(t) \leq c_2 g(t) \ \text{ for } \ 0 < t < \varepsilon.$$

To avoid misunderstandings, it is convenient to mention what is going on in the cases excluded from the assertion above:

Remark 2.2 1. *If $s > \frac{n}{p}$, or if $s = \frac{n}{p}$ and $q \leq 1$ (in the case of B-spaces) or $p \leq 1$ (in the case of F-spaces), the estimate for the growth envelope function has no interest, because in these cases we have that the spaces are continuously embedded in L_∞, and therefore the growth envelope functions remain bounded.*

2. *The situation is different when $s \leq n(\frac{1}{p} - 1)_+$: if this inequality is strict, the function space in question contains other distributions in addition to the regular ones (it is not contained in L_1^{loc}), hence tools like the decreasing rearrangement do not make sense; in the case when $s = n(\frac{1}{p} - 1)_+$, then the existence of inclusion in L_1^{loc} depends on the values of the parameters*

p and q and, in fact, Haroske obtained also in [17] results which extend the above mentioned ones. We will, however, tend to exclude this so-called borderline *situation from our considerations; one reason is that for the so-called* fine index *— with which we shall briefly deal in the last part of this lecture — there is still no final results in this borderline situation, even for the classical Besov and Triebel-Lizorkin spaces.*

We illustrate now the results of Proposition 2.1 in the particular context of Sobolev spaces W_2^1, using the fact that they coincide (up to equivalent norms) with $F_{2,2}^1$: therefore,

Corollary 2.3 *Always near 0,*

1. $\mathcal{E}_{LG}|W_2^1(t) \sim t^{\frac{1}{n}-\frac{1}{2}}$ *if* $n > 2$;
2. $\mathcal{E}_{LG}|W_2^1(t) \sim |\log t|^{\frac{1}{2}}$ *if* $n = 2$;
3. $\mathcal{E}_{LG}|W_2^1(t) \sim 1$ *if* $n = 1$.

This corollary constitutes a good opportunity to relate this type of results with older ones, well-known to the mathematical analyst: using the fact that the embedding $X_1 \hookrightarrow X_2$ implies the inequality $\mathcal{E}_{\mathrm{LG}}|X_1 \leq \mathcal{E}_{\mathrm{LG}}|X_2$, we have that

- the upper estimate in assertion 1 of the above corollary is a consequence of the Sobolev embedding Theorem (which, in the present case, guarantees that $W_2^1 \hookrightarrow L_{\frac{2n}{n-2}}$);
- on the other hand, the corresponding lower estimate implies that $W_2^1 \not\hookrightarrow L_p$, $\forall p > \frac{2n}{n-2}$;
- in an analogous way, the lower estimate in assertion 2 of the above corollary (where — we recall — one is assuming $n = 2$) implies that $W_2^1 \not\hookrightarrow L_\infty$;
- while the corresponding upper estimate is a consequence of a famous result of Trudinger, about the embedding of W_2^1 into an Orlicz space.

The tools used in the proof of Proposition 2.1 were, in the subcritical case, interpolation theory (for the upper estimates) and construction of extremal functions (for the lower estimates), while in the critical case atomic decompositions and, again, extremals functions were respectively used.

In order to present results for more general spaces, we need to give here the definitions of these less known spaces. Partly by reasons of simplification of the presentation, and partly also due to the fact that, at the moment, the available results are not so complete for the spaces of type F, from now on we deal essentially only with spaces of Besov type.

Definition 2.4 *Given $s \in \mathbb{R}$, $0 < p,q \leq \infty$ and a monotone function $\Psi : (0,1] \to \mathbb{R}^+$ satisfying the relation*

$$\Psi(2^{-j}) \sim \Psi(2^{-2j}), \quad j \in \mathbb{N},$$

we define

$$B^{(s,\Psi)}_{p,q} := \{ f \in \mathcal{S}' : \| f | B^{(s,\Psi)}_{p,q} \| := \Big(\sum_{j=0}^{\infty} 2^{jsq} \Psi(2^{-j})^q \| (\varphi_j \hat{f})\check{} | L^p \|^q \Big)^{\frac{1}{q}} < \infty \}$$

(obvious modification for the case $q = \infty$),

where $\mathcal{S}'$ refers to the space of tempered distributions, $\hat{}$ and $\check{}$ stand, respectively, for the Fourier transform and its inverse, and $(\varphi_j)_{j \in \mathbb{N}_0}$ is the dyadic partition of unity usually considered when defining Besov spaces.

We obtain the classical Besov spaces if one considers $\Psi \equiv 1$. In general, the presence of Ψ induces a perturbation in the smoothness given by the parameter s. A typical example of such a Ψ is $\Psi(x) = \Psi_b(x) = (1 + |\log x|)^b$, for real fixed b. While the influence of the main smoothness s in the definition above is translated by the power 2^{js}, the additional influence of the function Ψ is given by a factor of logarithmic type.

In this form, the spaces from Definition 2.4 were introduced by Edmunds and Triebel in [11], [12], but in fact they belong to the class of the so-called Besov spaces of generalized smoothness considered since many years by the Russian school, in particular by Kalyabin and Goldman. More exact references can be seen in [14].

In 2001 and 2002, Caetano and Moura [7], [6] proved the following (and a corresponding result for spaces of type F):

Proposition 2.5 *Assuming that $s > n(\frac{1}{p} - 1)_+$,*

1. *if $s < \frac{n}{p}$ (the subcritical case),*

$$\mathcal{E}_{LG} | B^{(s,\Psi)}_{p,q}(t) \sim t^{\frac{s}{n} - \frac{1}{p}} \Psi(t)^{-1} \quad \textit{near } 0;$$

2. *if $s = \frac{n}{p}$ (the critical case), and assuming further that $(\Psi(2^{-j})^{-1})_j \notin \ell_{q'}$,*

$$\mathcal{E}_{LG} | B^{(\frac{n}{p},\Psi)}_{p,q}(t) \sim \Big(\int_{t^{1/n}}^{1} \Psi(y)^{-q'} \frac{dy}{y} \Big)^{\frac{1}{q'}} \quad \textit{near 0},$$

   where in the subcase $q \leq 1$ the conjugate q' is taken to be ∞ and the right-hand side of the equivalence above should be interpreted as $\sup_{t^{1/n} \leq y \leq 1} \Psi(y)^{-1}$.

We make here a remark corresponding to part 1 of Remark 2.2:

Remark 2.6 *If $s > \frac{n}{p}$, or if $s = \frac{n}{p}$ and $(\Psi(2^{-j})^{-1})_j \in \ell_{q'}$, the upper estimate for the growth envelope function has no interest, because in these cases one has that the spaces are continuously embedded in L_∞, and therefore the growth envelope functions remain bounded.*

As before, the tools used in the proof of Proposition 2.5 were, in the subcritical case, interpolation theory — now with a function parameter — (for the upper estimates) and the construction of extremal functions (for the lower estimates), while in the critical case atomic decompositions in the meantime obtained by Moura in [21] and, again, extremal functions were respectively used. And, thought the arguments had become substantially more complicated in comparison with the classical situation, the fact we have adopted a more abstract point of view allowed us to notice that the expression obtained in the critical case was suggesting a generalization that would unify the two cases (critical and subcritical). Summing up, in [6] it was already possible to state that, even if with different (though equivalent) expressions, we have the following unifying result (and a corresponding one for spaces of type F):

Proposition 2.7 *Assuming that $s > n(\frac{1}{p}-1)_+$ and $(2^{-j(s-\frac{n}{p})}\Psi(2^{-j})^{-1})_j \notin \ell_{q'}$, it holds*

$$\mathcal{E}_{LG}|B^{(s,\Psi)}_{p,q}(t) \sim \left(\int_{t^{1/n}}^{1} y^{(s-\frac{n}{p})q'}\,\Psi(y)^{-q'}\,\frac{dy}{y}\right)^{\frac{1}{q'}} \quad \textit{near 0},$$

using, in the subcase $q \le 1$, the same interpretations as before.

This remark was going to do us a great service afterwards, as we will see. Meanwhile there had been, influenced by [7], developments in what concerns the subcritical case in the context of function spaces of generalized smoothness. Interest in this type of spaces was reappearing. We have already referred the works of Edmunds and Triebel [11], [12] in the study of the spectral theory for isotropic fractal drums, but interest in such spaces was also arising for example in stochastics. More exact references can be found in the paper [14] of Farkas and Leopold. The point of view adopted by these authors is general enough to include the spaces considered by Goldman and Kalyabin (cf., for example, [15], [20]). However they were only able to get atomic representations (an important detail in our context) with some further restrictions. These we include already in the definition we give here (as before, we will only detail the case of spaces of Besov type) :

Definition 2.8 *Let $N := (N_j)_{j\in\mathbb{N}_0}$ and $\sigma := (\sigma_j)_{j\in\mathbb{N}_0}$ be two sequences of positive numbers such that there exist $c_0, c_1 > 0$ and $\lambda_0, \lambda_1 > 1$ for which one has that*

$$\forall j \in \mathbb{N}_0, \quad c_0\sigma_j \le \sigma_{j+1} \le c_1\sigma_j, \quad \lambda_0 N_j \le N_{j+1} \le \lambda_1 N_j.$$

Let $\kappa_0 \in \mathbb{N}$ be such that, for any $j \in \mathbb{N}_0$, $2N_j \le N_{j+\kappa_0}$.
Given $0 < p,q \le \infty$, define

$$B^{\sigma,N}_{p,q} := \{ f \in \mathcal{S}' : \| f | B^{\sigma,N}_{p,q} \| := \Big(\sum_{j=0}^{\infty} \sigma_j^q \| (\varphi_j^N \hat{f})\check{} | L_p \|^q \Big)^{\frac{1}{q}} < \infty \}$$

(obvious modification for the case $q = \infty$),

where $\mathcal{S}'$, $\hat{}$, $\check{}$ are as in Definition 2.4 and $(\varphi_j^N)_{j\in\mathbb{N}_0}$ is a sequence of infinitely differentiable nonnegative functions verifying the following conditions:

1. $\operatorname{supp} \varphi_j^N \subset \{\xi \in \mathbb{R}^n : |\xi| \le N_{j+\kappa_0}\}$ *if $j = 0, 1, \ldots, \kappa_0 - 1$;*

   $\operatorname{supp} \varphi_j^N \subset \{\xi \in \mathbb{R}^n : N_{j-\kappa_0} \le |\xi| \le N_{j+\kappa_0}\}$ *if $j = \kappa_0, \kappa_0 + 1, \ldots$;*

2. *for each $\gamma \in \mathbb{N}_0$, there is a constant $c_\gamma > 0$ such that*

$$|D^\gamma \varphi_j^N(\xi)| \le c_\gamma (1 + |\xi|^2)^{-\gamma/2} \quad \forall j \in \mathbb{N}_0, \quad \forall \xi \in \mathbb{R}^n;$$

3. *there is a constant $c_\varphi > 0$ such that*

$$0 < \sum_{j=0}^{\infty} \varphi_j^N(\xi) = c_\varphi < \infty, \quad \forall \xi \in \mathbb{R}^n.$$

Remark 2.9 *In the case when one considers $N_j = 2^j$ and $\sigma_j = 2^{js}\Psi(2^{-j})$, $j \in \mathbb{N}_0$ in the preceding definition, one obtains the space $B^{(s,\Psi)}_{p,q}$ from Definition 2.4. It is known that the conditions imposed on σ in Definition 2.8 are much more flexible than considering just a σ of the form pointed out in this remark. In particular, there might not exist a well identified exponent s related with σ. For more information, see section 2.2 of [14]*

By default, the general conditions to impose on N and σ in this lecture are the ones indicated in Definition 2.8. We need, moreover (for the result which follows), the following definition:

Definition 2.10 *With the notation*

$$\underline{\sigma}_l := \inf_{k\geq 0} \frac{\sigma_{l+k}}{\sigma_k} \quad \textit{and} \quad \overline{\sigma}_l := \sup_{k\geq 0} \frac{\sigma_{l+k}}{\sigma_k}, \quad l \in \mathbb{N}_0, \tag{2}$$

we define the upper and lower Boyd indices of σ *respectively by*

$$\alpha_\sigma = \lim_{l\to\infty} \frac{\log_2 \overline{\sigma}_l}{l} \quad \textit{and} \quad \beta_\sigma = \lim_{l\to\infty} \frac{\log_2 \underline{\sigma}_l}{l}. \tag{3}$$

Bricchi and Moura [2] proved in 2002 the following in the subcritical case (and a corresponding result for space of type F):

Proposition 2.11 *Assuming* $\beta_\sigma > n(\frac{1}{p}-1)_+$ *and* $\alpha_\sigma < \frac{n}{p}$*, it holds that*

$$\mathcal{E}_{LG}|B^{\sigma,(2^j)_j}_{p,q}(t) \sim t^{-\frac{1}{p}}\Lambda(t^{-\frac{1}{n}})^{-1} \quad \textit{near 0},$$

for any continuous function $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying the relationships* $\Lambda(bz) \sim \Lambda(z)$ *(with constants which might depend on b, but not on z),* $\Lambda(z^{-1}) = \Lambda(z)^{-1}$ *and* $\Lambda(z) \sim \sigma_j$ *for* $z \in [2^j, 2^{j+1}]$ *(constants independent of j).*

The tools used in the proof of this proposition were interpolation theory with a function parameter for the upper estimates and construction of extremal functions for the lower estimates, having for such effect benefited from the atomic representations obtained meanwhile by Farkas and Leopold in [14].

When $\sigma_j = 2^{js}$, $j \in \mathbb{N}_0$, then $\alpha_\sigma = \beta_\sigma = s$, and this justifies calling subcritical to the case treated in Proposition 2.11.

Still in the case $\sigma_j = 2^{js}$, if for each small positive t we can find j such that $2^{-(j+1)n} < t \leq 2^{-jn}$, that is, $2^{j+1} > t^{-\frac{1}{n}} \geq 2^j$, in such a way that $\Lambda(t^{-\frac{1}{n}}) \sim \sigma_j$ and

$$t^{-\frac{1}{p}}\Lambda(t^{-\frac{1}{n}})^{-1} \sim 2^{j\frac{n}{p}}\sigma_j^{-1} = 2^{j(-s+\frac{n}{p})} \sim t^{\frac{s}{n}-\frac{1}{p}},$$

we obtain, in fact, the behaviour near zero already established before in the classical context.

Hence, in order to generalize it seems convenient to detach in $t^{\frac{s}{n}-\frac{1}{p}}$ smoothness and integrability as $t^{-\frac{1}{p}}t^{\frac{s}{n}}$. By analogy, the formulæof Caetano and Moura [7] and of Bricchi and Moura [2] for the subcritical case are more directly comparable if one writes the former as $t^{-\frac{1}{p}}\,(t^{\frac{s}{n}}\Psi(t)^{-1})$.

After these remarks — and comparing with the unified form given in Proposition 2.7 — the following result recently proved by Caetano and Farkas [3] should not be totally surprising:

Proposition 2.12 *Assuming*

$$\begin{cases} (\underline{\sigma}_l^{-1})_{l\in\mathbb{N}_0} \in \ell_{\min(q,1)} & \text{if} \quad p>1 \\ (\underline{\sigma}_l^{-1}\overline{N}_l^{n(\frac{1}{p}-1)+\delta})_{l\in\mathbb{N}_0} \in \ell_{\min(q,1)}, \quad \text{for some} \quad \delta>0, & \text{if} \quad p\leq 1 \end{cases} \tag{4}$$

and

$$(\sigma_j^{-1}N_j^{\frac{n}{p}})_{j\in\mathbb{N}_0} \notin \ell_{q'}, \tag{5}$$

it holds that

$$\mathcal{E}_{LG}|B^{\sigma,N}_{p,q}(t) \sim \Big(\int_{t^{1/n}}^1 y^{-\frac{n}{p}q'}\Lambda(y^{-1})^{-q'}\frac{dy}{y}\Big)^{\frac{1}{q'}} \quad \text{near 0}$$

(usual modification for the case $q'=\infty$, i.e., when $q\leq 1$),

for any continuous function $\Lambda:\mathbb{R}^+\to\mathbb{R}^+$ satisfying the relationships $\Lambda(bz)\sim\Lambda(z)$ (with constants which might depend on b, but not on z) and $\Lambda(z)\sim\sigma_j$ for $z\in[N_j,N_{j+1}]$ (constants independent of j).

Similarly as in Remark 2.2, it is convenient to justify here the reason why some cases were excluded in the assertion above:

Remark 2.13

1. *If (5) is false, i.e., if $(\sigma_j^{-1}N_j^{\frac{n}{p}})_{j\in\mathbb{N}_0}\in\ell_{q'}$, the estimate of the growth envelope function has no interest, because in these cases the spaces are continuously embedded in L_∞, and therefore the growth envelope functions remain bounded.*

2. *The reasons for the requirement (4) are partly technical. Even so, it should be mentioned that (4) implies that*

$$(\sigma_j^{-1}N_j^{n(\frac{1}{p}-1)_+})_{j\in\mathbb{N}_0}\in\ell_{q'}, \tag{6}$$

   and that this is a sufficient condition for the function spaces under consideration to contain only regular distributions, what justifies the use of tools like the decreasing rearrangement. Nevertheless, even the latter condition is not necessary for the inclusion of the spaces in L_1^{loc}, so that it remains to be clarified what happens in the borderline *situation when* (6) *is false but, even so, the distributions under consideration are in L_1^{loc}. The justification for calling the latter a* borderline *situation comes from the comparison with the classical setting, since in that context the condition (4) is equivalent to the condition $s>n(\frac{1}{p}-1)_+$.*

The result presented in Proposition 2.12 includes, in particular, all the previously known results involving the estimation of growth envelope functions of spaces of Besov type, apart from borderline situations. The basic tool used in its proof was the atomic representation proved by Farkas and Leopold [14] for the spaces in question, together with the ideas which had already led Caetano and Moura to the proof of Proposition 2.7. In particular, the statement of Proposition 2.12 does not make any distinction between the critical case and the subcritical case.

We would also like to mention that recently Gurka and Opic [16] recovered some of the results proved in [7] and [6] without using atomic representations (nor interpolation theory). Instead they have strongly based their approach on the consideration of Hardy type inequalities.

Although after some time we have concentrated our attention on results for spaces of type B, we have pointed out in several occasions that there exist counterparts for spaces of type F. In fact, the bulk of the work is generally done for the first type of spaces, being possible in most cases to study the other type of spaces by some method of reduction to what was obtained for spaces of type B. Unfortunately, it is not so clear how such a reduction can be done starting from Proposition 2.12, so that at the time when the work [3] was complete it was not known how to prove corresponding results for spaces of type F. Recently, however, in collaboration with H.-G. Leopold, it seems we have found out a way of solving the problem, but this joint work is still under discussion.

3 Complements

3.1 The fine index

In the preceding section we choose, for ease of exposition, to report only on results for growth envelope functions. However, and in accordance to what was mentioned in the Introduction, the concept of growth envelope involves, besides the growth envelope function, also the concept of fine index. It is fairly more complicated to compute such index, by comparison with the estimation of the growth envelope function, and also we don't want to go here through the history of the various results that had been obtained until one reaches the most general one. Therefore we state only the result of Caetano and Farkas in [3], which was obtained after several partial generalizations starting from the fundamental works [17] of Haroske and [25] of Triebel.

Nevertheless, we need to introduce some further notation first:

- we denote by $\Phi(t)$ the expression $\left(\int_{t^{1/n}}^{1} y^{-\frac{n}{p}q'} \Lambda(y^{-1})^{-q'} \frac{dy}{y}\right)^{\frac{1}{q'}}$ (or the corresponding one in the case $q' = \infty$) considered in Proposition 2.12;

- we denote by μ the Borel measure associated with $-\log_2 \Phi$ in $(0,\varepsilon]$, for any $\varepsilon \in (0,1)$, i.e., the only Borel measure μ in $(0,\varepsilon]$ such that $\mu([a,b]) = -\log_2 \Phi(b) - (-\log_2 \Phi(a)) = \log_2 \frac{\Phi(a)}{\Phi(b)}$, $\forall\, [a,b] \subset (0,\varepsilon]$.

We are now ready to state the general result of Caetano and Farkas [3]:

Proposition 3.1 *Under the same conditions as in Proposition 2.12, it holds that the least $v > 0$ for which there are $\varepsilon > 0$ small enough and $c(v) > 0$ such that*

$$\left(\int_0^\varepsilon \left(\frac{f^*(t)}{\Phi(t)} \right)^v \mu(dt) \right)^{\frac{1}{v}} \leq c(v) \| f | B^{\sigma,N}_{p,q} \| \tag{7}$$

for all $f \in B^{\sigma,N}_{p,q}$ exists and is equal to q.

Remark 3.2 *In the case $v = \infty$, the left-hand side of* (7) *must be read as*

$$\sup_{t \in (0,\varepsilon]} \frac{f^*(t)}{\Phi(t)}.$$

It is even possible to prove that the inequality (7) does not stand if one substitutes $\frac{f^*(t)}{\Phi(t)}$ by $\kappa(t)\frac{f^*(t)}{\Phi(t)}$, for any given positive function κ decreasing in $(0,\varepsilon]$, unless κ is chosen bounded.

The measure μ can in several situations be determined explicitly. For example, if $q > 1$ then the expression $\mu(dt)$ in (7) can be replaced by

$$\frac{dt}{\Phi(t)^{q'}\, t^{\frac{q'}{p}}\, \Lambda(t^{-\frac{1}{n}})^{q'}\, t}.$$

3.2 Continuity envelopes

As was mentioned several times, when the function spaces are in L_∞, there is no interest to estimate the growth envelope function, because this one is then always bounded. Furthermore, at least in the context of Besov and Triebel-Lizorkin spaces, the embedding in L_∞ ensures that all the functions from these spaces are uniformly continuous. One can, nevertheless, ask about how far from being Lipschitzian is the "worst" function of such a space.

In many respects one can develop a theory and results parallel with what was described for growth envelopes. The basic tool now, instead of the decreasing rearrangement of functions, is the modulus of continuity, or, to be more precise, the quotient between modulus of continuity and the independent variable. From that starting point one builds the concept of continuity envelope function $\mathcal{E}_{\mathrm{C}}|X$ of the function space X by passing to the supremum over normalized functions. In mathematical notation,

$$\mathcal{E}_{\mathrm{C}}|X(t) := \sup_{\|f|X\|\le 1} \frac{\omega(f,t)}{t}, \quad t>0,$$

where $\omega(f,t) = \sup_{|h|\le t} \sup_{x\in\mathbb{R}^n} |f(x+h)-f(x)|, \quad t>0.$

We are not going into details. As in the preceding case, there are connections with results well-known to the mathematical analyst, as is the case of the famous result of Brézis and Wainger [1] about the "almost" Lipschitz continuity of the functions from the Sobolev space $W_p^{1+\frac{n}{p}}$, $1<p<\infty$. However, the notion itself was only introduced by Haroske in [17], under influence of the works [9] and [10] of Edmunds and Haroske, and the first results in the new language were obtained by Haroske and Triebel in [17] and [25]. For the more recent results, involving spaces $B_{p,q}^{(s,\Psi)}$ (in accordance with Definition 2.4) and $F_{p,q}^{(s,\Psi)}$, see [19] and [4].

3.3 Applications

Due to its characteristic of searching for the "worst" possible behaviour, growth envelope and continuity envelope functions are specially well adapted to the task of obtaining necessary conditions for the existence of embeddings between function spaces. For example, it was in this way that we proved in [3] that the condition $(\sigma_j^{-1} N_j^{\frac{n}{p}})_{j\in\mathbb{N}_0} \in \ell_{q'}$ is necessary for the existence of the embedding $B_{p,q}^{\sigma,N} \hookrightarrow L_\infty$ (or, what is equivalent in this case, for the existence of the embedding $B_{p,q}^{\sigma,N} \hookrightarrow C$, where C denotes the space of bounded and uniformly continuous functions endowed with the supremum norm), after we have proved in [6] an analogous result in the context of the spaces $B_{p,q}^{(s,\Psi)}$. Since in the same works we had already proved that the conditions in question were sufficient, we obtained, in this way, conditions which are simultaneously necessary and sufficient for the existence of those continuous embeddings. This makes clear that growth envelope functions are indeed an adequate tool for this type of task.

Another application was the proof, in [4], that, in the case $s_1 \ge s_2$, $0<p_1\le p_2\le\infty$, with $s_1-\frac{n}{p_1} = s_2-\frac{n}{p_2}$, the condition

$$\left(\frac{\Psi_2\left(2^{-j}\right)}{\Psi_1\left(2^{-j}\right)}\right)_{j\in\mathbb{N}} \in \ell_{q^*}, \quad \text{where } q^* \text{ is given by } \quad \frac{1}{q^*} := \left(\frac{1}{q_2}-\frac{1}{q_1}\right)_+,$$

which had already, in [21], been proved to be sufficient for the existence of the embedding

$$B^{(s_1,\Psi_1)}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{(s_2,\Psi_2)}_{p_2,q_2}(\mathbb{R}^n),$$

is also necessary for the existence of such embedding. In [4] our option was to use as tool the concept of continuity envelope in the proof, due to the fact that work is about that type of envelopes. It is important to highlight that, in this case, besides the envelope function, also the fine index performed a fundamental role.

There is also a connection between envelope functions and approximation numbers a_k, $k \in \mathbb{N}$. Such a connection was, for example, explored in [5] in order to obtain sharp estimates for approximation numbers of embeddings between function spaces by means of the knowledge of the growth envelope functions of the spaces involved. Though it is not always possible to obtain sharp estimates, it is amazing that it is indeed possible in some cases. And, even if in [5] it was the growth envelope function which was used as a tool, in this context it is more natural to use the continuity envelope function. In fact, as pointed out in [17] and [4], the following direct result comes from an estimate obtained already some years ago by Carl and Stephani [8, Thm. 5.6.1]:

Proposition 3.3 *Let $X(U)$ be a Banach space of functions defined in the unit ball U in $\mathbb{R}^n$ with $X(U) \hookrightarrow C(U)$. There exists $c > 0$ such that, for all $k \in \mathbb{N}$,*

$$a_k\,(id : X(U) \longrightarrow C(U)) \le c\,k^{-\frac{1}{n}}\,\mathcal{E}_C|X(k^{-\frac{1}{n}})\,.$$

References

[1] H. Brézis and S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Comm. Partial Differential Equations **5** (1980), 773–789.

[2] M. Bricchi and S. D. Moura, *Complements on growth envelopes of spaces with generalized smoothness in the sub-critical case*, Z. Anal. Anwend. **22** (2003), no. 2, 383–398.

[3] A. M. Caetano and W. Farkas, *Local growth envelopes of Besov spaces of generalized smoothness*, preprint CM 04/I-24, Univ. Aveiro, 2004.

[4] A. M. Caetano and D. D. Haroske, *Continuity envelopes of spaces of generalised smoothness: a limiting case; embeddings and approximation numbers*, J. Function Spaces Appl., to appear.

[5] ______, *Sharp estimates of approximation numbers via growth envelopes*, Function Spaces, Differential Operators and Nonlinear Analysis. The Hans

Triebel Anniversary Volume (Basel) (D. Haroske, T. Runst, and H.-J. Schmeisser, eds.), Birkhäuser, 2003, pp. 237–244.

[6] A. M. Caetano and S. D. Moura, *Local growth envelopes of spaces of generalized smoothness: the critical case*, to appear in Math. Ineq. & Appl.

[7] ______, *Local growth envelopes of spaces of generalized smoothness: the sub-critical case*, Math. Nachr. **273** (2004), no. 1, 43–57.

[8] B. Carl and I. Stephani, *Entropy, compactness and the approximation of operators*, Cambridge University Press, 1990.

[9] D. E. Edmunds and D. D. Haroske, *Spaces of Lipschitz type, embeddings and entropy numbers*, Dissertationes Math. **380** (1999), 1–43.

[10] ______, *Embeddings in spaces of Lipschitz type, entropy and approximation numbers, and applications*, J. Approx. Theory **104** (2000), no. 2, 226–271.

[11] D. E. Edmunds and H. Triebel, *Spectral theory for isotropic fractal drums*, C. R. Acad. Sci. Paris **326(I)** (1998), 1269–1274.

[12] ______, *Eigenfrequencies of isotropic fractal drums*, Oper. Theory Adv. Appl. **110** (1999), 81–102.

[13] ______, *Sharp Sobolev embeddings and related Hardy inequalities: the critical case*, Math. Nachr. **207** (1999), 79–92.

[14] W. Farkas and H.-G. Leopold, *Characterisations of function spaces of generalised smoothness*, Annali di Matematica pura ed Applicata, to appear.

[15] M. L. Goldman, *A method of coverings for describing general spaces of Besov type*, Trudy Mat. Inst. Steklov **156** (1980), 47–81, (Russian) [English transl.: Proc. Steklov Inst. Math. **156** (2) (1983)].

[16] P. Gurka and B. Opic, *Sharp embeddings of Besov spaces with logarithmic smoothness*, preprint.

[17] D. D. Haroske, *Envelopes in function spaces - a first approach*, preprint Math/Inf/16/01, Univ. Jena, Germany, 2001.

[18] ______, *Limiting embeddings, entropy numbers and envelopes in function spaces*, Habilitationsschrift, Institute of Mathematics of Friedrich-Schiller University Jena, 2002.

[19] D. D. Haroske and S. D. Moura, *Continuity envelopes of spaces of generalized smoothness, entropy and approximation numbers*, J. Approximation Theory **128** (2004), no. 2, 151–174.

[20] G. A. Kalyabin, *Description of functions in classes of Besov-Lizorkin-Triebel type*, Trudy Mat. Inst. Steklov **156** (1980), 82–109, (Russian) [English transl.: Proc. Steklov Inst. Math. **156** (2) (1983)].

[21] S. Moura, *Function spaces of generalised smoothness*, Dissertationes Math. **398** (2001), 88 pp.

[22] Yu. Netrusov, *Embedding theorems of Besov spaces into ideal spaces*, Notes of Scientific Seminar LOMI **159** (1987), 69–82, (Russian) [English translation: J. Soviet. Math. **47** (1989), 2871-2881].

[23] ______, *Sets of singularities of functions in spaces of Besov and Lizorkin-Triebel type*, Trudy Mat. Inst. Steklov **187** (1989), 162–177, (Russian) [English translation: Proc. Steklov Inst. Math. **187** (1990), 185-203].

[24] H. Triebel, *Sharp Sobolev embeddings and related Hardy inequalities: the sub-critical case*, Math. Nachr. **208** (1999), 167–178.

[25] ______, *The structure of functions*, Birkhäuser, Basel, 2001.