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On the type of convergence in atomic representations

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Dedicated to Professor V. Burenkov on the occasion of his 70th birthday

Abstract

The type of convergence in atomic representations of spaces of Besov and Triebel-Lizorkin is usually presented in the sense of the topology of the tempered distributions, occasionally with some remarks about the possibility of the convergence being valid in some Lebesgue spaces, if some conditions are met. Until now we are not aware of any explicit indication that those representations usually converge in the Besov or Triebel-Lizorkin spaces themselves. Yet this is indeed the case, as explained in this note.

We deal also with a corresponding question for wavelet representations in a recently introduced class of generalized local Hardy spaces.

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1 Introduction

Results on atomic representations of spaces of Besov and Triebel-Lizorkin, either of classical or generalized smoothness, are always presented, as far as we are aware, in terms of convergence in the sense of tempered distributions (see, for example, [6] and [7]). Sometimes a remark is added about the possibility of the convergence being valid in some Lebesgue spaces, under some restrictions on the parameters. However, it follows from the structure of the assertions on the atomic representations that, possibly apart from the cases when the parameters are infinite, the convergence holds (even unconditionally) in the spaces themselves. One of the main aims of this note is to present straighforward proofs that this is indeed the case. The other aim is to prove something similar regarding wavelet representations in a class of generalized local Hardy spaces recently introduced (see [1]), leading us to show that they admit unconditional Schauder bases formed by wavelets. We deal with the first question in section 2 and with second one in section 3.

2 Atomic representations in Besov and Triebel-Lizorkin spaces.

We start by recalling several ingredients we shall need, following [6], [7]. The first one is the meaning of the atoms to be considered.

By $Q_{\nu m}$, with $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, we denote the closed cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-\nu}m$ and with side length $2^{-\nu+1}$. Given a cube Q in $\mathbb{R}^n$and $r > 0$, by rQ we denote the cube in $\mathbb{R}^n$ which is concentric with Q and has side length r times the side length of Q.

Definition 2.1. Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $K, L \in \mathbb{N}_0$ and $c \geq 1$. A continuous function $a_{\nu m} : \mathbb{R}^n \to \mathbb{C}$, with $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, for which there exist all (classical) derivatives $D^\alpha a_{\nu m}$ if $|\alpha| \leq K$ is called a $(s,p)_{K,L,c}$-atom (or, briefly, a (s,p)-atom) if

$$\operatorname{supp} a_{\nu m} \subset cQ_{\nu m},$$
$$|D^\alpha a_{\nu m}(x)| \leq 2^{-\nu(s-n/p)+\nu|\alpha|} \qquad \text{for } |\alpha| \leq K$$

and

$$\int_{\mathbb{R}^n} x^\beta a_{\nu m}(x)\, dx = 0 \qquad \text{when } \nu \neq 0 \text{ and } \beta \in \mathbb{N}_0^n \text{ with } |\beta| < L. \tag{1}$$

Clearly, the condition (1) is interpreted as non-existing when $L = 0$ is taken. Note also that the cancellation in (1) is not required when $\nu = 0$.

Next we recall the sequence spaces in which the coefficients in the atomic representations shall live, for which we need the following notation:

Given $0 < p \leq \infty$, $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, $\chi^{(p)}_{\nu m}(x) := 2^{(\nu-1)n/p}$ if $x \in Q_{\nu m}$ and $\chi^{(p)}_{\nu m}(x) := 0$ if $x \notin Q_{\nu m}$.

Definition 2.2. Let $0 < p, q \leq \infty$ and write $\lambda := (\lambda_{\nu m})_{\substack{\nu \in \mathbb{N}_0 \\ m \in \mathbb{Z}^n}}$, where $\lambda_{\nu m}$ are complex numbers.

(i) b_{pq} is the quasi-Banach space of those (generalized) sequences λ for which

$$\|\lambda | b_{pq}\| := \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q}$$

(with the usual modification if p or q are ∞) is finite.

(ii) f_{pq} is the quasi-Banach space of those (generalized) sequences λ for which

$$\|\lambda | f_{pq}\| := \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m} \chi^{(p)}_{\nu m}(\cdot)|^q \right)^{1/q} | L_p(\mathbb{R}^n) \right\|$$

(with the usual modification if q is ∞) is finite.

Finally, we recall the atomic representation theorem for the Besov spaces $B^s_{pq}(\mathbb{R}^n)$ and the Triebel-Lizorkin spaces $F^s_{pq}(\mathbb{R}^n)$. These are the usual spaces defined by using Fourier-analytical tools — see, for example, [6, Definition 1.2 (pp. 4-5)].

We use, as usual, the notation $\sigma_p := n(\frac{1}{p} - 1)_+$ and $\sigma_{pq} := n(\frac{1}{\min\{p,q\}} - 1)_+$.

Theorem 2.3. *(i) Let $0 < p, q \le \infty$, $s \in \mathbb{R}$, $K, L \in \mathbb{N}_0$ and $c \ge 1$ satisfy*

$$K > s \qquad \text{and} \qquad L > \sigma_p - s.$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}, \qquad \text{unconditional convergence in } \mathcal{S}'(\mathbb{R}^n), \tag{2}$$

where $a_{\nu m}$ are (s,p)-atoms and $\lambda \in b_{pq}$. Furthermore,

$$\|f|B^s_{pq}(\mathbb{R}^n)\| \approx \inf \|\lambda|b_{pq}\|$$

are equivalent quasi-norms, where the infimum is taken over all admissible representations (2).

(ii) Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, $K, L \in \mathbb{N}_0$ and $c \ge 1$ satisfy

$$K > s \qquad \text{and} \qquad L > \sigma_{pq} - s.$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $F^s_{pq}(\mathbb{R}^n)$ if, and only if, it can represented as in (2), *where $a_{\nu m}$ are (s,p)-atoms and $\lambda \in f_{pq}$. Furthermore,*

$$\|f|F^s_{pq}(\mathbb{R}^n)\| \approx \inf \|\lambda|f_{pq}\|$$

are equivalent quasi-norms, where the infimum is taken over all admissible representations (2).

Remark 2.4. The (unconditional) convergence in $\mathcal{S}'(\mathbb{R}^n)$ of the sum in (2) is not an assumption: it follows from the assumptions on λ and the hypotheses on the parameters.

We are now ready to state and prove one of the main results in this note:

Theorem 2.5. *Let $0 < p, q < \infty$ and $s \in \mathbb{R}$. Any atomic representation of a given distribution f in $B^s_{pq}(\mathbb{R}^n)$ or $F^s_{pq}(\mathbb{R}^n)$, according to the theorem above, converges unconditionally in $B^s_{pq}(\mathbb{R}^n)$ or $F^s_{pq}(\mathbb{R}^n)$ respectively.*

Proof. (i) Consider first the case of $f \in B^s_{pq}(\mathbb{R}^n)$ and a representation like (2). Since, given any $T \in \mathbb{N}$,

$$\left(\sum_{\nu=0}^{T} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q} \le \|\lambda|b_{pq}\|,$$

then also the partial sums

$$\sum_{\nu=0}^{T} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}$$

belong to $B^s_{pq}(\mathbb{R}^n)$ and, moreover, again using the theorem above,

$$\begin{aligned} \left\| f - \sum_{\nu=0}^{T} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} | B^s_{pq}(\mathbb{R}^n) \right\| &= \left\| \sum_{\nu=T+1}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} | B^s_{pq}(\mathbb{R}^n) \right\| \\ &\lesssim \left(\sum_{\nu=T+1}^{\infty} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q}, \end{aligned}$$

which tends to 0 as T goes to ∞. This proves the convergence to f in $B^s_{pq}(\mathbb{R}^n)$ of the sum in (2) using the order that it exhibits. In order to ensure the unconditional convergence in the same space, we prove the summability of the sum to f, i.e., that

$$\begin{aligned} &\forall \delta > 0, \exists \mathcal{N}_0 \subset \mathbb{N}_0 \times \mathbb{Z}^n \text{ with } \sharp\mathcal{N}_0 < \infty : \\ &\quad \forall \mathcal{N} \in \mathbb{N}_0 \times \mathbb{Z}^n \text{ with } \sharp\mathcal{N} < \infty, \quad \mathcal{N} \supset \mathcal{N}_0 \Rightarrow \| f - \sum_{(\nu,m)\in\mathcal{N}} \lambda_{\nu m} a_{\nu m} | B^s_{pq}(\mathbb{R}^n) \| < \delta. \end{aligned}$$

In fact, let $\delta > 0$ be given and consider $\mathcal{N}_0 := \{(\nu, m) \in \mathbb{N}_0 \times \mathbb{Z}^n : \nu \leq \nu_0,\ |m| \leq m_0\}$, with $\nu_0, m_0 \in \mathbb{N}$ to be chosen depending on δ. Then, for $\mathcal{N} \supset \mathcal{N}_0$ we can write, with the first sums converging unconditionally in $\mathcal{S}'(\mathbb{R}^n)$,

$$\begin{aligned} &\| f - \sum_{(\nu,m)\in\mathcal{N}} \lambda_{\nu m} a_{\nu m} | B^s_{pq}(\mathbb{R}^n) \| \\ &= \| \sum_{(\nu,m)\notin\mathcal{N}} \lambda_{\nu m} a_{\nu m} | B^s_{pq}(\mathbb{R}^n) \| \\ &\lesssim \| \sum_{\nu=0}^{\nu_0} \sum_{\substack{m\in\mathbb{Z}^n \text{s.t.} \\ (\nu,m)\notin\mathcal{N}}} \lambda_{\nu m} a_{\nu m} | B^s_{pq}(\mathbb{R}^n) \| + \| \sum_{\nu=\nu_0+1}^{\infty} \sum_{\substack{m\in\mathbb{Z}^n \text{s.t.} \\ (\nu,m)\notin\mathcal{N}}} \lambda_{\nu m} a_{\nu m} | B^s_{pq}(\mathbb{R}^n) \| \\ &\lesssim \left(\sum_{\nu=0}^{\nu_0} \left(\sum_{|m|>m_0} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q} + \left(\sum_{\nu=\nu_0+1}^{\infty} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q}. \end{aligned}$$

We have already seen above that the second term tends to 0 when ν_0 goes to ∞, therefore it is possible to choose $\nu_0 \in \mathbb{N}$ such that this second term is dominated by a suitable constant times δ. As to the first term, from the hypothesis $\|\lambda | b_{pq}\| < \infty$ it follows that, for each $\nu \in 0, \ldots, \nu_0$, we can choose $m(\nu) \in \mathbb{N}$ such that

$$\left(\sum_{|m|>m(\nu)} |\lambda_{\nu m}|^p \right)^{q/p} < c_1 \frac{\delta^q}{\nu_0 + 1},$$

where c_1 is a suitably chosen positive constant; choosing then $m_0 := \max_{\nu=0,\ldots,\nu_0} m(\nu)$ we arrive at

$$\left(\sum_{\nu=0}^{\nu_0}\left(\sum_{|m|>m_0}|\lambda_{\nu m}|^p\right)^{q/p}\right)^{1/q} \le \left(\sum_{\nu=0}^{\nu_0} c_1\frac{\delta^q}{\nu_0+1}\right)^{1/q} = c_1^{1/q}\delta.$$

(ii) Consider now the case $f \in F^s_{pq}(\mathbb{R}^n)$ and a representation like (2). Similarly as in part (i), given any $T \in \mathbb{N}$, we arrive at

$$\begin{aligned}\left\|f - \sum_{\nu=0}^{T}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}|F^s_{pq}(\mathbb{R}^n)\right\| &= \left\|\sum_{\nu=T+1}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}|F^s_{pq}(\mathbb{R}^n)\right\| \\ &\lesssim \left\|\left(\sum_{\nu=T+1}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}\chi^{(p)}_{\nu m}(\cdot)|^q\right)^{1/q}|L_p(\mathbb{R}^n)\right\|,\end{aligned}$$

and the conclusion that this tends to 0 as T goes to ∞ follows from the Lebesgue dominated convergence theorem. This proves the convergence to f in $F^s_{pq}(\mathbb{R}^n)$ of the sum in (2) using the order that it exhibits. In order to ensure the unconditional convergence in the same space, we do as in part (i), proving the summability of the sum to f, i.e., that

$$\begin{aligned}&\forall\delta>0, \exists\mathcal{N}_0 \subset \mathbb{N}_0\times\mathbb{Z}^n \text{ with } \sharp\mathcal{N}_0<\infty : \\ &\quad\forall\mathcal{N}\in\mathbb{N}_0\times\mathbb{Z}^n \text{ with } \sharp\mathcal{N}<\infty,\quad \mathcal{N}\supset\mathcal{N}_0 \Rightarrow \|f-\sum_{(\nu,m)\in\mathcal{N}}\lambda_{\nu m}a_{\nu m}|F^s_{pq}(\mathbb{R}^n)\|<\delta.\end{aligned}$$

We proceed as in part (i), considering $\delta > 0$, $\mathcal{N}_0$ chosen of the same type, with $\nu_0, m_0 \in \mathbb{N}$ to be chosen depending on δ, and $\mathcal{N} \supset \mathcal{N}_0$. Similarly, we arrive at

$$\begin{aligned}&\|f-\sum_{(\nu,m)\in\mathcal{N}}\lambda_{\nu m}a_{\nu m}|F^s_{pq}(\mathbb{R}^n)\| \\ &\lesssim \left\|\left(\sum_{\nu=0}^{\nu_0}\sum_{|m|>m_0}|\lambda_{\nu m}\chi^{(p)}_{\nu m}(\cdot)|^q\right)^{1/q}|L_p(\mathbb{R}^n)\right\| \\ &\quad+\left\|\left(\sum_{\nu=\nu_0+1}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}\chi^{(p)}_{\nu m}(\cdot)|^q\right)^{1/q}|L_p(\mathbb{R}^n)\right\|.\end{aligned} \tag{3}$$

We have already seen above that the last term tends to 0 when ν_0 goes to ∞, therefore it is possible to choose $\nu_0 \in \mathbb{N}$ such that this second term is dominated by a suitable constant times δ. However, the same type of argument — starting by writing $\sum_{\nu=0}^{\nu_0}\sum_{|m|>m_0}|\lambda_{\nu m}\chi^{(p)}_{\nu m}(\cdot)|^q$ as $\sum_{|m|>m_0}\sum_{\nu=0}^{\nu_0}|\lambda_{\nu m}\chi^{(p)}_{\nu m}(\cdot)|^q$, upper estimating this by $\sum_{|m|>m_0}\sum_{\nu=0}^{\infty}|\lambda_{\nu m}\chi^{(p)}_{\nu m}(\cdot)|^q$ and choosing an ordering of the

m's which goes from smaller $|m|$'s to larger ones — also allow us to control the term (3): we just have to choose m_0 such that

$$\left\| \sum_{|m|>m_0} \sum_{\nu=0}^{\infty} |\lambda_{\nu m} \chi_{\nu m}^{(p)}(\cdot)|^q |L_p(\mathbb{R}^n) \right\| \leq c_2 \delta,$$

where c_2 is a suitably chosen positive constant. □

If we conjugate the just proved theorem with some well-known embedding properties of the Besov and the Triebel-Lizorkin spaces, we get in a simple way results of convergence of atomic representations in more classical spaces (without claiming that this is easier than a direct proof). We present two examples, assuming in both that $0 < p, q < \infty$, in order that Theorem 2.5 can be applied:

Since, for $s > \sigma_p$, $B^s_{pq}(\mathbb{R}^n), F^s_{pq}(\mathbb{R}^n) \hookrightarrow L_{\bar{p}}(\mathbb{R}^n)$, where $\bar{p} := \max\{1, p\}$, then in such a case any atomic representation of a function f in those Besov or Triebel-Lizorkin spaces converges also unconditionally to f in $L_{\bar{p}}(\mathbb{R}^n)$. And since, for $s > n/p$, $B^s_{pq}(\mathbb{R}^n), F^s_{pq}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n)$, where $C(\mathbb{R}^n)$ stands for the space of complex-valued, bounded and uniformly continuous functions on $\mathbb{R}^n$ endowed with the sup norm, then in such a case any atomic representation of a function f in those Besov or Triebel-Lizorkin spaces converges also unconditionally and uniformly to f.

Remark 2.6. Looking at the structure of the atomic representation theorems for Besov and Triebel-Lizorkin spaces of generalized smoothness as considered in [5] (see also [4]) and [2], it is clear that a result corresponding to Theorem 2.5 also holds for such spaces of generalized smoothness.

3 Unconditional Schauder bases of wavelets in generalized Hardy spaces

Such bases have been obtained in many function spaces, for example in the Besov and Triebel-Lizorkin spaces we considered in the previous section. Recently, a class of spaces called generalized Hardy spaces have been introduced for which wavelet representations were derived — see [1]. However, with respect to the existence of corresponding unconditional Schauder bases, in the latter paper it was only announced that it would be proved elsewhere. We deal with that matter here.

We need to recall a few things.

As to the system $\Psi_k := (\psi^{j,G}_m)_{j,G,m}$, for k large enough, of wavelets we shall consider, we refer, for details, to [6] or to the just mentioned paper [1]. Here we just mention that they are real compactly supported wavelets of Daubechies (inhomogeneous) type of class C^k, with vanishing moments until order k, given by

$$\psi^{j,G}_m(x) := \begin{cases} \psi^G_m(x) & j = 0,\ G \in G^0,\ m \in \mathbb{Z}^n \\ 2^{\frac{j-1}{2}n}\, \psi^G_m(2^{j-1}x) & j \in \mathbb{N},\ G \in G^j,\ m \in \mathbb{Z}^n \end{cases},$$

where G^j denotes sets of n-tuples which count the possible combinations of basic *father* and *mother* wavelets to be considered. And, for simplicity, we shall omit the set of indices from the notation.

The generalized Hardy spaces we shall consider here are defined by

$$h_q(\phi) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f \,|\, h_q(\phi)\|_\varphi := \left\|\mathcal{R}^1_\varphi f \,|\, \Lambda_q(\phi)\right\| < \infty \right\},$$

where

- $0 < q < \infty$;
- $\phi : (0, \infty) \to (0, \infty)$ is continuous with $\phi(1) = 1$ and $\bar{\phi}(t) := \sup_{s>0} \frac{\phi(st)}{\phi(s)}$ is finite for every $t > 0$; and such that $\beta_{\bar{\phi}} := \lim_{t\to+\infty} \frac{\log \bar{\phi}(t)}{\log t} > 0$;
- $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$;
- $(\mathcal{R}^1_\varphi f)(x) := \sup_{0<t<1} |(\varphi_t * f)(x)|$, $x \in \mathbb{R}^n$, with $\varphi_t(x) := t^{-n}\varphi(x/t)$;
- $\Lambda_q(\phi)$ is the generalized Lorentz space defined as the collection of all measurable functions f on $\mathbb{R}^n$ such that

  $$\|f|\Lambda_q(\phi)\| := \left(\int_0^\infty [f^*(t)\phi(t)]^q \, \frac{dt}{t} \right)^{1/q} < \infty,$$

  with f^* the usual decreasing rearrangement of f.

For further details and considerations, we refer to [1]. In particular, the definition of $h_q(\phi)$ is independent of the φ considered, in the sense of equivalent quasi-norms, and therefore we shall simply write $\|f \,|\, h_q(\phi)\|$ instead of $\|f \,|\, h_q(\phi)\|_\varphi$.

We need also the definition of the sequence spaces $\lambda_q(\phi)$, with q and ϕ as above:

$$\lambda_q(\phi) := \{\mu \equiv (\mu^{j,G}_m)_{j,G,m} \subset \mathbb{C} : \ \|\mu|\lambda_q(\phi)\| < \infty\},$$

where

$$\|\mu|\lambda_q(\phi)\| := \left\| \left(\sum_{j,G,m} |\mu^{j,G}_m \chi_{jm}(\cdot)|^2 \right)^{1/2} |\Lambda_q(\phi) \right\|,$$

with χ_{jm} the characteristic function of the cube Q_{jm} introduced in the beginning of the preceding section.

In [1] the following result was proved:

Theorem 3.1. *Let q, ϕ and Ψ_k be as just introduced. There exists $k(\phi) \in \mathbb{N}$ such that, for every $\mathbb{N} \ni k > k(\phi)$, the following holds: $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $h_q(\phi)$ if, and only if, it can be represented as*

$$f = \sum_{j,G,m} \mu^{j,G}_m \, 2^{-jn/2} \psi^{j,G}_m \quad \textit{with} \quad \mu = (\mu^{j,G}_m)_{j,G,m} \in \lambda_q(\phi) \tag{4}$$

(summability in $\mathcal{S}'(\mathbb{R}^n)$). Moreover, the wavelet coefficients $\mu_m^{j,G}$ are uniquely determined by

$$\mu_m^{j,G} = \mu_m^{j,G}(f) := 2^{jn/2}\langle f, \psi_m^{j,G}\rangle.$$

Furthermore,

$$\|f \,|\, h_q(\phi)\| \approx \|\mu(f) \,|\, \lambda_q(\phi)\| \qquad \text{(equivalent quasi-norms)},$$

where $\mu(f) \equiv (\mu_m^{j,G}(f))_{j,G,m}$.

Remark 3.2. The summability in $\mathcal{S}'(\mathbb{R}^n)$ of the sum in (4) is not an assumption: it follows from the assumptions on μ and the hypotheses on the parameters.

We can finally state and prove the second main result of this note:

Theorem 3.3. *Under the hypotheses of the preceding theorem (namely with $k > k(\phi)$), the system Ψ_k is an unconditional Schauder basis in $h_q(\phi)$.*

Proof. From Theorem 3.1, any $f \in h_q(\phi)$ can be represented in a unique way as an infinite linear combination of the elements of Ψ_k, in the sense of summability in $\mathcal{S}'(\mathbb{R}^n)$. So, what remains to be proved is that such a linear combination is also summable in $h_q(\phi)$, as this implies summability in $\mathcal{S}'(\mathbb{R}^n)$ (this follows from the characterization of $h_q(\phi)$ as an interpolation space of classical Hardy spaces — cf. [1] — and the fact that the latter are continuously embedded in $\mathcal{S}'(\mathbb{R}^n)$).

We start as in the proof of Theorem 2.5. Since, given any $T \in \mathbb{N}$,

$$\left\| \left(\sum_{j=0}^{T} \sum_{G,m} |\mu_m^{j,G} \chi_{jm}(\cdot)|^2 \right)^{1/2} |\Lambda_q(\phi) \right\| \leq \|\mu|\lambda_q(\phi)\|,$$

then also the partial sums

$$\sum_{j=0}^{T} \sum_{G,m} \mu_m^{j,G}\, 2^{-jn/2} \psi_m^{j,G}$$

belong to $h_q(\phi)$ and, moreover, again using the theorem above, with the convergence of the first sums being in $\mathcal{S}'(\mathbb{R}^n)$,

$$\begin{aligned} &\left\| f - \sum_{j=0}^{T} \sum_{G,m} \mu_m^{j,G}\, 2^{-jn/2} \psi_m^{j,G} |h_q(\phi) \right\| \\ &= \left\| \sum_{j=T+1}^{\infty} \sum_{G,m} \mu_m^{j,G}\, 2^{-jn/2} \psi_m^{j,G} |h_q(\phi) \right\| \\ &\lesssim \left\| \left(\sum_{j=T+1}^{\infty} \sum_{G,m} |\mu_m^{j,G} \chi_{jm}(\cdot)|^2 \right)^{1/2} |\Lambda_q(\phi) \right\|. \end{aligned} \tag{5}$$

Notice now that from the hypothesis (and Theorem 3.1), we can write

$$\left(\int_0^\infty \left\{\left[\left(\sum_{j=0}^\infty \sum_{G,m} |\mu_m^{j,G} \chi_{jm}(\cdot)|^2\right)^{1/2}\right]^*(t)\, \phi(t)\right\}^q \frac{dt}{t}\right)^{1/q} < \infty,$$

from which follows that the expression inside the integral must be finite a.e., the same must be true to the decreasing rearrangement part (because $\phi > 0$), and therefore

$$\left(\sum_{j=0}^\infty \sum_{G,m} |\mu_m^{j,G} \chi_{jm}(\cdot)|^2\right)^{1/2}$$

is also finite a.e.. Consequently, the rest inside the quasi-norm in (5) must go to 0 a.e. as T tends to ∞. Now our hypotheses guarantee that the dominated convergence theorem for $\Lambda_q(\phi)$ — cf. [3, Proposition 2.3.3]) — can be applied, allowing us to conclude that (5) goes to 0 as T tends to ∞ and prove that

$$f = \sum_{j=0}^\infty \sum_{G,m} \mu_m^{j,G}\, 2^{-jn/2} \psi_m^{j,G} \qquad \text{in } h_q(\phi).$$

It only remains to show that this convergence is unconditional. This can be done in much the same way as in the summability part in part (ii) of the proof of Theorem 2.5. The fact that we have now the extra index G in the summation causes no big changes in the approach, as G has only a finite number of possibilities. Otherwise we are now considering $h_q(\phi)$ instead of $F^s_{pq}(\mathbb{R}^n)$ and $\Lambda_q(\phi)$ instead of $L_p(\mathbb{R}^n)$ (and some other minor modifications), so we skip the details. □

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