ASYMPTOTIC DISTRIBUTION OF WEYL NUMBERS AND EIGENVALUES

by

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To my parents,

who have given me the moments that should have been theirs

To Ana Leonor,

who has given me the moments we should have been sharing together

To Ana Carlota,

to whom I hope to give the moments that should be mine

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PREFACE

I hereby declare that this thesis has not been submitted, either in the same or different form, to this or any other University for a degree.

I also declare that, unless otherwise mentioned in the text (namely in the introductions), the results presented in this thesis come from the author's own original work.

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SUMMARY

Embeddings between function spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ are studied, where $s \in \mathbb{R}$, $p,q \in (0,\infty]$ and Ω is a bounded C^∞-domain, the order of decay of the corresponding Weyl numbers being determined. This was not known before in the case when at least one of p,q is less than 1. The technique used is discretization, reducing the task to the consideration of the corresponding problem for embeddings between sequence spaces. Accordingly, the estimates for the Weyl numbers of these embeddings are also new when at least one of the spaces present is not a Banach space, and, in order to deal with the latter problem, a new technique is developed, where we introduce a new kind of volume numbers; as a consequence, some new results about the volume of sections of unit balls in sequence spaces are also derived.

Some concrete examples of irregular domains are presented, for which the remainder in the asymptotic formula for the eigenvalues of the Dirichlet Laplacian exhibits a non-standard (non-power) behaviour. Disconnected domains are studied first, and then, by means of a Dirichlet-Neumann bracketing technique, connected domains are also dealt with. At the same time, comparison with the Minkowski dimensions of the boundaries of the domains in question is performed, and it turns out that this is insufficient to characterize the variety of possible forms of the remainders referred to above.

A family of operators modelled on the Stokes operator is also dealt with, and, for the case of periodic boundary conditions, a big o estimate is obtained for the remainder in the asymptotic formula for the corresponding eigenvalues. This result is also meant to be a first step in the study of the general problem.

TABLE OF NOTATION

GENERAL

		(absence of symbol between two objects) besides the more usual meanings associated with some sort of multiplication between elements of the same or different sets, and also for the action of an operator or matrix on an element of the domain or a vector, we also write $\alpha A := \{\alpha x : x \in A\}$ and $-A := -1A$ (where A is a set)
	but	see also III.2.3 for a different meaning of 2R, with R a set
$:=$		is equal, by definition, to
∂		boundary of
$\emptyset$		empty set
$\setminus$		set-theoretical difference
$\subset$		set-theoretical inclusion (allowing for equality)
	or	inclusion of a sequence in a set, meaning then that the elements of the sequence belong to the set
^{c}		complement of (example: ^cA is the complement of the set A)
#		cardinal of (example: #A is the cardinal of the set A),
	but	see also IV.2.1 and IV.2.2 for a meaning associated with periodic functions
∇		gradient of
	but	see also special notation $(\nabla\cdot,\nabla\cdot)_\Omega$ in (I.4.5)
$\lhd$		is a closed subspace of — I.2.5.2
+		operation in an additive group
	or	sum of subsets of an additive group (example: $A+B := \{x+y : x \in A, y \in B\}$)
	or	non-negative part of (example: $f_+ := \max\{f,0\}$, with f a real function)
	or	subset of (strictly) positive numbers of a set, as in $\mathbb{R}^+$
$\overline{}$		complex conjugate of (example: $\bar{a}$ is the complex conjugate of $a \in \mathbb{C}$)
	or	closure of (example: $\overline{A}$ is the closure of the subset A of a topological space),
	but	can also have any suitable ephemeral meaning
$*$		convolution of functions or distributions
	or	adjoint of (example: T^* is the adjoint of the operator or the

		matrix T)
	but	see also $B^*(\xi)$ in IV.2.4
$\wedge$		Fourier transform of (example: $f^\wedge$ or $\hat{f}$ is the Fourier transform of the function or distribution f) — I.3.2; the definition adopted is, in the case of a function f, $f^\wedge := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(y)\, e^{-i \cdot y}\, dy$, $n \in \mathbb{N}$
$\vee$		inverse Fourier transform of (example: $f^\vee$ or $\check{f}$ is the inverse Fourier transform of the function or distribution f) — I.3.2; see also the preceding notation
$\vert$		restriction to (example: $f\vert_\Omega$ is the restriction of the function or distribution f to the set Ω)
$\sim$		used in $f(x) \sim g(x)$ as $x \to l$ means that $\lim_{x \to l} f(x)/g(x) = 1$, where f,g are real functions
$\approx$		used in $f(x) \approx g(x)$ as $x \to l$ means that there exist $c_1, c_2 > 0$ such that $c_1 f(x) \le g(x) \le c_2 f(x)$ for all x close enough to l, where f,g are real functions
	or	used in $f(k,M) \approx g(k,M)$ as $k,M \to \infty$ means that there exist c_1, c_2, k_0, $M_0 > 0$ such that $k \ge k_0, M \ge M_0 \Rightarrow c_1 g(k,M) \le f(k,M) \le c_2 g(k,M)$, where f,g are real functions, $k,M \in \mathbb{N}$
$1\!\!1_A(x)$		$\begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$
$\succ$		is more peaked than — I.1.4
$\times$		product of measures
	or	Cartesian product
	or	symbol separating the number of rows and columns of a matrix
$\cdot$		symbol taking the place of a variable in a function (note that it is a point positioned slightly above the current line)
.		operation in a multiplicative group
	or	scalar product
	or	usual inner product in $\mathbb{R}^n$ or $\mathbb{C}^n$ ($n \in \mathbb{N}$)
$(\cdot,\cdot)_X$		inner product in the space X
$(\nabla\cdot,\nabla\cdot)_\Omega$		(I.4.5)
(V,H,b)		variational triplet — I.4.1 — , where V, H are Hilbert spaces and b is a form
$(\cdot,\cdot)$		open interval
	or	ordered pair

$]\cdot,\cdot[$ $[\cdot,\cdot)$ $(\cdot,\cdot]$ $[\cdot,\cdot]$	usual open, half-open and closed intervals
$[x]$	greatest integer $\leq x$ $(x\in\mathbb{R})$ **or** reference x
$[(x_j)_{j=1}^n]$	$([x_j])_{j=1}^n$ $((x_j)_{j=1}^n\in\mathbb{R}^n,\ n\in\mathbb{N})$
$(x_j)_{j=1}^n$ $(x_j)_{j=1,\dots,n}$	ordered n-tuple $(n\in\mathbb{N}\cup\{\infty\})$
x^β	$x_1^{\beta_1}\cdot\ldots\cdot x_n^{\beta_n}$ $(x:=(x_j)_{j=1}^n\in\mathbb{R}^n$ or $\mathbb{C}^n$, $\beta:=(\beta_j)_{j=1}^n\in\mathbb{N}_o^n$, $n\in\mathbb{N})$
$(x_j)_j$	finite or infinite sequence or ordered -tuple (j runs over a set understood from the context)
$\{x_j\}_j$	set of indexed elements (j runs over a set understood from the context)
$\lVert\cdot\rVert$	quasi-norm in general — I.2.1 — **or** operator quasi-norm — (I.2.4)
$\lVert\cdot\vert X\rVert$ $\lVert\cdot\rVert_X$	quasi-norm in the space X
$\vert\cdot\vert$	modulus in $\mathbb{R}$ or $\mathbb{C}$ **but** in Chapter III stands for $\vert\cdot\vert_2$
$\lVert f\rVert_p$	$\left(\int_{\mathbb{R}^n}\vert f(x)\vert^p\,dx\right)^{1/p}$ if $p\in(0,\infty)$; $\operatorname{ess\,sup}_{x\in\mathbb{R}^n}\vert f(x)\vert$ if $p=\infty$ (f real or complex on $\mathbb{R}^n$, $n\in\mathbb{N}$) — I.3.1(i)
$\vert(x_j)_j\vert_p$	$\left(\sum_j\vert x_j\vert^p\right)^{1/p}$ if $p\in(0,\infty)$; $\sup_j\vert x_j\vert$ if $p=\infty$ $(x_j\in\mathbb{R}$ or $\mathbb{C})$
a. e.	almost everywhere
$a_k(\cdot)$	approximation numbers — I.2.5.1 **but** a_k (not a function) can also be used with other meanings
as $x\to 1$	besides the usual meanings related to limits and in $\sim$, $\approx$, little o and big o, used in C(x) as $x\to 1$, where C(x) is a condition involving the variable x, means that C(x) is true for all x close enough to 1 — III.3.1
$B_X(d,r)$	$\{x\in X: \lVert x-d\rVert_X\leq r\}$ $(d\in X,\ r\geq 0)$, **but** see (III.4.1) for B(a,l) and IV.1.1 for $B(x,\xi)$
$\overset{o}{B}_X(d,r)$	interior of $B_X(d,r)$

B_X	$B_X(0,1)$
$\overset{o}{B}_X$	interior of B_X
$B_p^M(d,r)$	$B_{l_p^M}(d,r)$ ($d\in l_p^M$, $r\geq 0$, $M\in\mathbb{N}$, $p\in(0,\infty]$)
$B_p(d,r)$	$B_{l_p}(d,r)$ ($d\in l_p$, $r\geq 0$, $p\in(0,\infty]$)
$\overset{o}{B}{}_p^M(d,r)$	interior of $B_p^M(d,r)$ ($d\in l_p^M$, $r\geq 0$, $M\in\mathbb{N}$, $p\in(0,\infty]$)
$\overset{o}{B}_p(d,r)$	interior of $B_p(d,r)$ ($d\in l_p$, $r\geq 0$, $p\in(0,\infty]$)
B_p^M	$B_p^M(0,1)$ ($M\in\mathbb{N}$, $p\in(0,\infty]$)
B_p	$B_p(0,1)$ ($p\in(0,\infty]$)
$\overset{o}{B}{}_p^M$	interior of B_p^M ($M\in\mathbb{N}$, $p\in(0,\infty]$)
$\overset{o}{B}_p$	interior of B_p ($p\in(0,\infty]$)
$B_{pq}^s(\mathbb{R}^n)$	I.3.3(i)
$B_{pq}^s(\Omega)$	I.3.5(i)
$c_k(\cdot)$	Gelfand numbers — I.2.5.2
but	c_k (not a function) can also be used with other meanings
$\text{codim}_V E$	$\dim \frac{V}{E}$
$C_o^m(\Omega)$	set of m-times continuously differentiable functions defined on Ω and having compact support contained in Ω ($m\in\mathbb{N}_o\cup\{\infty\}$, Ω is a non-empty open subset of $\mathbb{R}^n$, $n\in\mathbb{N}$; 0-times continuously differentiable just means continuous)
$C_S^\infty(\mathbb{R}^n)$, $C_S^\infty(\Omega)$	I.4.13
$C_S^m(\mathbb{R}^n)$, $C_S^m(\Omega)$	same definition of the preceding notation except that ∞ is replaced by $m\in\mathbb{N}_o$
$C_S(\Omega)$	$C_S^o(\Omega)$
$C_\#^\infty(\overset{o}{B}{}_\infty^n(d,\delta/2))$	IV.2.1
det	determinant of matrix or operator
$d_\infty(\cdot,\cdot)$	distance corresponding to $\lvert\cdot\rvert_\infty$-norm
dist $(\cdot,\cdot)$	distance corresponding to the Euclidean norm $\lvert\cdot\rvert_2$
$d_k(S_b,H)$	I.4.4

$D'(\Omega)$	I.3.1(iv)
$\mathcal{D}(T)$	domain of operator T — I.2.1
D_j	partial derivative with respect to the variable in coordinate j
D_j^k	k-th partial derivative with respect to the variable in coordinate j
D^α	$D_1^{\alpha_1} \circ \dots \circ D_n^{\alpha_n}$ $(\alpha=(\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n,\ n \in \mathbb{N})$
but	see III.3.2(ii) for the meaning of D^B with B a set
δ_{jm}	Kronecker symbol
δ_j^M	$(\delta_{jm})_{m=1}^M$ — I.2.8
$\Delta_h^M \varphi$	$\sum_{l=0}^{M} \varphi(\cdot + lh) \binom{M}{l} (-1)^{M-1}$ $(h \in \mathbb{R}^n,\ M,n \in \mathbb{N},\ \varphi$ function on $\mathbb{R}^n)$
$-\Delta$	$-\sum_{j=1}^{n} D_j^2$ $(n \in \mathbb{N})$
$-\Delta_S^\Omega$	I.4.15 — with Ω, S sets
$F_{pq}^s(\mathbb{R}^n)$	I.3.3(ii)
$F_{pq}^s(\Omega)$	I.3.5(ii)
$\Gamma(x)$	Euler gamma function: $\int_0^\infty e^{-y} y^{x-1} dy$, $x>0$
but	see a temporary meaning of Γ as a set in I.3.12
$H^s(\Omega)$	$W_2^s(\Omega)$, Sobolev space
$H_0^s(\Omega)$	closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$
$H_S^1(\Omega)$	closure of $C_S^\infty(\Omega)$ in $H^1(\Omega)$
$H_\#^m(\mathring{B}_\infty^n(d,\delta/2))$	IV.2.1
i	$\sqrt{-1}$
or	index
iff	if and only if
$I_q^p(M)$	natural embedding $l_p^M \longrightarrow l_q^M$
Id_X	identity map $X \longrightarrow X$
but	see also Id_j in II.4.6
and	Id in IV.2.7
$\mathbb{K}^M$	I.2.8

$\mathcal{L}(X,Y)$	I.2.1
l_p^M	real or complex space of M-tuples equipped with the quasi-norm $\lvert\cdot\rvert_p$ ($M\in\mathbb{N}\cup\{\infty\}$, $p\in(0,\infty]$)
l_p	l_p^∞ ($p\in(0,\infty]$)
$L_p(\mathbb{R}^n)$	I.3.1(i)
$L_p^Q(\mathbb{R}^n)$	I.3.1(v)
$N(\cdot,V,H,b)$	I.4.2 – where V, H are Hilbert spaces and b is a form
$\mathbb{N}$	set of integers ≥ 1
$\mathbb{N}_o$	$\mathbb{N}\cup\{0\}$
O	big o (example: $f(x)=g(x)+O(h(x))$ as $x\to 1$ means that there exists $c>0$ such that $\lvert f(x)-g(x)\rvert\leq c\,h(x)$ for all x close enough to 1, where f,g,h are real functions)
o	little o (example: $f(x)=o(g(x))$ as $x\to 1$ means that $\lim_{x\to 1} f(x)/g(x)=0$, where f,g are real functions)
$\prod_{j=1}^{m} b_j$	I.4.5 – where b_j are forms, $m\in\mathbb{N}$
$\mathbb{R}^+$	$\{x\in\mathbb{R} : x>0\}$
$\mathbb{R}^-$	$\{x\in\mathbb{R} : x<0\}$
$\mathcal{R}(T)$	range of operator T
s	s-function – I.2.2 – **but** s (in normal typeface) can also have other meanings
$s_k(\cdot)$	k-th s-number (for a general s-function) – I.2.2 – **but** s_k (not a function) can also have other meanings – cf. I.3 and II.4.1
S^{n-1}	∂B_2^n ($n\in\mathbb{N}$)
$S(\mathbb{R}^n)$	I.3.1(ii)
$S'(\mathbb{R}^n)$	I.3.1(iii)
supp	support of
U_M	I.1.3 **but** see IV.2.5 for the meaning of $U_j(\nu)$

$v_k(\cdot)$	volume numbers — I.2.10
$\mathrm{Vol}_M(\cdot)$	outer, Lebesgue-Borel or Lebesgue measure in $\mathbb{R}^M$ or $\mathbb{C}^M$, $M\in\mathbb{N}$ — cf. I.2.8
$\mathrm{Vol}_H(\cdot)$	outer, Lebesgue-Borel or Lebesgue measure in the space H — see I.2.8

$\mathrm{Vol}_M(B_p^M) = \dfrac{(2\Gamma(1+1/p))^M}{\Gamma(1+M/p)}$ ($p\in(0,\infty]$, $M\in\mathbb{N}$, real B_p^M); moreover, there exists $\Theta:(0,\infty)\to\mathbb{R}$ such that $0<\Theta(t)<1/12$, $t>0$, and for any $p\in(0,\infty)$

$$\mathrm{Vol}_M(B_p^M) = 2^M\left(\frac{2\pi}{p}\right)^{(M-1)/2} M^{-M/p-1/2} e^{pM\Theta(1/p)-\Theta(M/p)p/M}$$ —

see [Edmunds/Triebel 1989, (3.1/1), (3.1/2)]

$W_p^s(\Omega)$	Sobolev space ($s\in\mathbb{N}_0$, $p\in(1,\infty)$)
$x_k(\cdot)$	Weyl numbers — I.2.5.3 — **but** x_k (not a function) can also be used with other meanings

SECTION I.1

$\succ$	I.1.4
A^y	$\{x\in\mathbb{R}^M : (x,y)\in A\}$, $y\in\mathbb{R}^N$, $M,N\in\mathbb{N}$, $A\subset\mathbb{R}^M\times\mathbb{R}^N$ — I.1.8
$\mathcal{B}_M$	I.1.1
K_s	I.1.1
λ_K	I.1.1
$\mathcal{P}_M$	I.1.1
t	I.1.1
u_M	I.1.3
Vol_M	I.1.1
w.r.t.	with respect to — I.1.1

SECTION I.2

I_k	natural identification $\mathbb{R}^{2k}\longrightarrow\mathbb{C}^k$, $k\in\mathbb{N}$ — I.2.8

SECTION I.3

f_N	(I.3.5)
f^N	(I.3.6)
p, p_1, p_2	$\in(0,\infty]$ — I.3
q, q_1, q_2	$\in(0,\infty]$ — I.3
s, s_1, s_2	$\in\mathbb{R}$ — I.3
Ω	I.3

SECTION I.4

I	used in $T-\xi I$, with $\xi\in\mathbb{C}$ and T an (unbounded) operator in a Hilbert space H, means $Id_H\|_{\mathcal{D}(T)}$
S_b	I.4.4 — where b is a form

SECTION II.4

c(N,j)	(II.4.12)
$\delta := s_1-s_2-n(1/p_1-1/p_2)_+ > 0$ — II.4.1	
E	(II.4.8)
E_{111}	see II.4.1
$F^N, F_{N,2}$	(II.4.9)
$F_{N,1}$	II.4.3
F_j	(II.4.11), II.4.7 — where $j\in\{H, \ldots, N\}$
F_{H-1}	II.4.6, (II.4.19)
$F_{N,1}(\Omega)$, $F_{N,2}(\Omega)$, $F^N(\Omega)$	II.4.3, II.4.4
$F_j(\Omega)$	II.4.6, II.4.7 — where $j\in\{H, \ldots, N\}$
$F_{H-1}(\Omega)$	II.4.6, (II.4.15)
F^{111}_{222}	this can appear instead of F in some the above notation — see II.4.1 for the meaning
φ_j	I.3.2

SECTION II.2

$H(\varepsilon)$	II.2.2
J_k	I.2.8
μ_p^M	II.2.6
R, R_k	I.2.8
U_M	I.1.3
Vol_H	I.2.8

SECTION II.4 (continuation)

$I(\Omega)$, $I^{s_1p_1q_1}_{s_2p_2q_2}(\Omega)$, $I^{111}_{222}(\Omega)$	II.4.1
Id_j	II.4.6
$J_r(\Omega)$	II.4.2
$J_{111,r}(\Omega)$	see II.4.1
M_j	(II.4.11) — where $j\in\{H, \ldots, N\}$
p, p_1, p_2	$\in(0,\infty]$ — II.4.1
q, q_1, q_2	$\in(0,\infty]$ — II.4.1
ψ	I.3.13 (satisfying (I.3.8))
ψ_λ	I.3.13
R	(II.4.8)
R_{111}	see II.4.1
s, s_1, s_2	$\in\mathbb{R}$ — II.4.1
S_j	(II.4.11) — where $j\in\{H, \ldots, N\}$
T_j	(II.4.11) — where $j\in\{H, \ldots, N\}$
Ω	II.4.1

CHAPTER III

$\|\cdot\|$	$\|\cdot\|_2$ — III.1.2
∂B_ε	(III.3.1)
2R	III.2.3 — different from usual meaning
$\mathcal{A}$	III.3.9
α	$\log_x \frac{a(x)}{h(x)}$ (<0) when $a\in\mathcal{A}$ is given — III.3.9
B	III.1.2 **but** see special meaning in III.4.2
B(a)	III.3.3
B(a,l)	(III.4.1)
B(a,-1/2)	B(a)
D, D^B	III.3.2 — where B is a set
f(x)	$-\alpha x^{-1}a(x)$ when $a\in\mathcal{A}$ is given — III.3.9
$\mathcal{H}$	III.3.6
h(x)	$x^{-\alpha}a(x)$ when $a\in\mathcal{A}$ is given — III.3.9
h'	derivative of h
λ_j	I.4.11
$\log^N$	look just before III.3.16
M_d^B	III.3.2
$N(\cdot)$, $N^B(\cdot)$	(III.1.3)
$Vol(\cdot)$	$Vol_2(\cdot)$ — III.1.2
R(a,i)	III.3.3
$Z(\cdot)$, $Z^B(\cdot)$	(III.1.4)

SECTION IV.2

$a, a_{p,q}, a_{p,q}^{\alpha,\beta}$	IV.2.4
a_ξ	IV.2.6
$A(\xi), A_{p,q}(\xi)$	IV.2.4
α	$\in\mathbb{N}_o^n$
$B, B_{j,p}, b_{j,p}^\alpha$	IV.2.3
$B(\xi), b_{j,p}(\xi)$	IV.2.3
$B^*(\xi)$	adjoint of $B(\xi)$
β	$\in\mathbb{N}_o^n$
c_0, c_1, c_2	respectively (IV.2.3), IV.2.10 and (IV.2.22)
C, C', C"	respectively (IV.2.22), (IV.2.7) and (IV.2.18)
C(k)	(IV.2.17)
d	centre of $\Omega = \overset{o}{B}{}_\infty^n(d,\delta/2)$ — IV.2.2
δ	side length of $\Omega = \overset{o}{B}{}_\infty^n(d,\delta/2)$ — IV.2.2
φ_ν	IV.2.5
Id	IV.2.7
J	IV.2.2
$L(\Omega)$	IV.2.2
$\lambda_k(\xi)$	IV.2.7 — where k=1, ..., N-J
$\Lambda(k), \Lambda(k)_\varepsilon$	IV.2.10
m, m_j	IV.2.2
μ	IV.2.8
N	IV.2.2
$N(\lambda,\xi)$	IV.2.8
$U_j(\nu), U(\nu)$	resp. IV.2.5 and (IV.2.6)
$V_\#(\Omega), V_\xi$	resp. IV.2.5 and IV.2.6
$W_\#(\Omega)$, Ω	IV.2.2
$X_\#(\Omega)$	IV.2.2

INTRODUCTION

The motto of this work is 'asymptotic distribution'.

Following this perspective, we study two kinds of numbers: Weyl numbers and eigenvalues. Both of them are associated to operators and, in fact, each operator (or, at least, those we are concerned with) has a sequence of either kind of numbers associated with it. The problem is then to study the properties of these sequences at infinity. More precisely, denoting by $(s_k)_{k\in\mathbb{N}}$ one of these sequences, our goal is to find known functions f of k which 'behave like' $k \longmapsto s_k$ and, in this sense, ours is a problem of modelling. This 'behaving like' can be understood in the sense of existence of constants $c_1, c_2, k_0 > 0$ such that

$$c_1 f(k) \leq s_k \leq c_2 f(k) \tag{1}$$

whenever $k \geq k_0$, which we shall abbreviate to $s_k \approx f(k)$ as $k \to +\infty$. Or sometimes we can be more precise and say that there exists $a \neq 0, \pm\infty$ such that

$$\lim_{k\to\infty} \frac{s_k}{f(k)} = a \ , \tag{2}$$

setting this as the definition of $s_k \sim a f(k)$ as $k \to +\infty$. These are examples of asymptotic distributions for $(s_k)_{k\in\mathbb{N}}$, and the latter situation is indeed the first step in the building up of an asymptotic formula for the sequence under study, for we can consider now $(s_k - af(k))_{k\in\mathbb{N}}$ and try to repeat the process. If we then can find g such that $s_k - af(k) \sim bg(k)$ as $k \to +\infty$, for some constant $b \neq 0, \pm\infty$, we will have found two terms in the asymptotic expansion for s_k, and it is in this sense that we shall write

$$s_k = af(k) + bg(k) + \ldots \tag{3}$$

In this work the farthest we can go is to the determination of g but not of b, that is, when trying to repeat the process we use $\approx$ instead of $\sim$ (as a matter of fact, it should be mentioned that there may not exist such a b in some cases).

In Chapter I we collect some background material, but there are some new things there also, in particular the definition of a new kind of volume numbers which will play an important role afterwards. The reader should therefore refer to the end of this introduction, where we make more detailed comments about that chapter.

In Chapter II we study the Weyl numbers of embedding operators between function spaces of the type $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$, with Ω an open bounded C^∞-domain, $s\in\mathbb{R}$, $p,q\in(0,\infty]$, obtaining estimates of the kind of (1). There are plenty of new results here, especially among those involving quasi-Banach spaces. Since our approach is through discretization, thus reducing the problem to the study of embeddings between sequence spaces, estimates of the kind of (1) are also presented for the Weyl numbers of the latter operators. Additionally, and as a natural by-product of the approach followed, there are also some new results concerning sections of unit balls in sequence spaces.

In Chapter III we present examples which show that g in (3) can assume a great many forms in the case when $(s_k)_{k\in\mathbb{N}}$ is the sequence of the eigenvalues of the Dirichlet Laplacian for *fractal* domains. We do not use this sequence directly in the estimates, but, instead, follow the current trend of studying the associated counting function $N(\lambda) := \sum_{s_k\le\lambda} 1$. There is no problem in following this approach, as the results obtained can be translated into more direct assertions involving the eigenvalues. They can also be translated into corresponding assertions for the partition function $Z(t) := \sum_{j=1}^{\infty} e^{-s_j t}$, $t>0$.

Chapter IV is also concerned with eigenvalues, but this time of a family of operators modelled on the Stokes operator. We also study the corresponding counting functions, and determine a big o estimate for a possible second term in the asymptotic expansion in the case of periodic boundary conditions. Furthermore, we make a conjecture for the general case of underlying Lipschitz domains.

In the introductions to the chapters after the first there are more detailed comments about what follows in the corresponding chapter. These introductions should also be checked, as well as the beginning of sections, for the notations and conventions in force in the corresponding subdivision of the thesis.

However, some notation might not be defined in the main text, but will, we believe, be in the Table of Notation provided. This has a general list for notation used coherently throughout the work and, where appropriate, special lists for some of the chapters and sections.

We also include an index which can, on occasions, convey information not appearing elsewhere in the text.

Any doubt about originality of some result should be checked against the introductions to the chapters (also the end of this introduction in the case of Chapter I) or on the spot. Absence of references to known work will then mean that the result in question either is ours or belongs to the mathematical folklore. Moreover, mention of a reference does not mean that the result was first proved there, especially if it is a book — the reader should consult the relevant reference to sort it out.

There should be no difficulty in understanding the cross references. Usually they consist of a group of numbers, enclosed between brackets only if they refer to a formula or diagram (in which case we should look through the right margins to find the result being referred to), the first number being either a Roman or an Arabic numeral. If it is a Roman numeral it refers to the chapter with that number; otherwise it refers to a section within the chapter where the reference is made. Cross referencing within the same subsection is sometimes simplified to a more down-to-earth system. The chapters should be easily identifiable by means of the heading running through the pages.

References to the bibliography is made between square brackets and using the surnames of the authors and the year of publication, sometimes with an additional letter if otherwise ambiguity could occur; material not yet published is referred by a letter in place of the year of publication. When the names of the authors are already clear, we do not repeat them inside the square brackets. These can also accomodate two or more references by the same authors at the same time, the different items being separated by semicolons.

We have tried to avoid the use of expressions like 'increasing' alone, preferring to use 'strictly increasing' or 'non-decreasing', which are more universally understood. We would, however, use 'increasing' and 'decreasing' in the broad sense, and 'positive' and 'negative' in the strict sense.

There is some abuse of language tacitly assumed. For example, given (y',y''), with $y' := (y_1, \dots, y_k)$, $y'' := (y_{k+1}, \dots, y_M)$, we identify (y',y'') with $y := (y_1, \dots, y_M)$; or regular distributions with the respective functions; or the elements of $L_p(\Omega)$ with functions in the respective equivalence classes; or matrices with the corresponding operators; and, though not always, write $f(x)$ for f.

Works dealing with asymptotic distributions usually have an inflation of constants throughout the exposition, and this is no exception. Though it may seem contradictory, constants may depend on several things, but not on the important ones. Here they will usually be denoted by lowercase c or uppercase c (in the latter case they tend to be more stable), most of the times with additional primes and subscripts to distinguish between them, but in some instances we shall also allow the, apparently, same c to denote different constants in two consecutive formulae.

We finish then with some comments about the contents of Chapter I.

The material of I.1 is taken from [Kanter 1977], with some modifications suggested by [Meyer/Pajor 1988]. We decided to include it here because it will be needed in Chapter II (more precisely, in the relevant section 2), and the way it was treated originally in [Kanter 1977] owes much to probabilistic concerns. Thus Kanter used some heavier machinery (such as probabilities defined on a space of probabilities) which is not at all necessary in the situations we are concerned with. Our exposition is therefore simpler and goes straighter towards our aims.

Section I.2 features some general definitions and properties which, however, would be difficult to find in textbook form. This is the case of the s-numbers in the context of quasi-Banach spaces (although for some facts about these spaces we could have referred to [Pietsch 1987, B.1]); of Lemma I.2.9 (the complex version of a well-known result for Lebesgue measure in $\mathbb{R}^n$ — we don't know who should have the credit for this, but it goes at least back to a paper by Carl and Triebel [1980, pp. 129-130]); and also of the (new) volume numbers, which will prove useful in the estimation of Weyl numbers in sequence spaces in II.3.

Section I.3 describes the spaces B^s_{pq} and F^s_{pq} and lists some of their properties. Though some of these are surely by now quite well-known and can be referred to Triebel's book [1983], we thought some room should be reserved to them in this work. After all, these spaces are the very *raison d'être* of Chapter II. Anyway, some other results stated in I.3 — those due to Edmunds and Triebel [1989; a] — are indeed quite recent. Our general policy with regard to proofs throughout all these reference material is not to write them down if they can be seen elsewhere. This does not imply, however, any claims for originality where the proofs are presented. Thus, we give a proof of Lemma I.3.10, but this statement had already been implicitly used by Edmunds and Triebel [1989, (4.3/44)]. As to Lemma I.3.19, we decided to present a sketch of the proof for the reader's convenience, since the paper [Edmunds/Triebel a] where it is originally proved is not yet published at the moment of writing this.

Finally, section I.4 sets the framework for the problems with eigenvalues to be discussed in Chapters III and IV. We found it more convenient to borrow the approach presented in [Métivier 1977], collecting in the way some results scattered in [Edmunds/Evans 1987]. The material starting with subsection I.4.13 aims to give a precise formulation to concepts such as "the mixed Dirichlet-Neumann boundary conditions for the Laplacian" in the weak sense, and Propositions I.4.17 and I.4.18 give some relations between counting functions associated with Laplacians having different such boundary conditions. Results of this kind seem to be known by everybody concerned with these matters, but we couldn't find any reference general enough to suit us (something in this direction, though only for the pure Dirichlet or Neumann boundary conditions, can be seen in [Reed/Simon 1978, pp. 269-270]).

CHAPTER I

PRELIMINARIES

1. Unimodality and peakedness of probability measures

1.1. Throughout this section $\mathcal{B}_M$ denotes the σ-field of the Borel subsets of $\mathbb{R}^M$, $M\in\mathbb{N}$, $\mathcal{P}_M$ denotes the collection of all probability measures defined on $\mathcal{B}_M$, and Vol_M stands for the Lebesgue-Borel measure on $\mathcal{B}_M$ (in integrals we shall write simply dx or dy instead of $d\mathrm{Vol}_M(x)$ or $d\mathrm{Vol}_M(y)$ respectively).

Given $K\in\mathcal{B}_M$ such that $\mathrm{Vol}_M(K)\in(0,\infty)$, we define $\lambda_K\in\mathcal{P}_M$ by

$$\lambda_K(A) = \frac{\mathrm{Vol}_M(A\cap K)}{\mathrm{Vol}_M(K)} \quad , \quad A\in\mathcal{B}_M \ .$$

Note that $\frac{1}{\mathrm{Vol}_M(K)}\mathbb{1}_K$ is a density of λ_K with respect to (w.r.t., for short) Vol_M.

Given $\mu\in\mathcal{P}_M$ with density f w.r.t. Vol_M, we define, for each $s>0$, $K_s=f^{-1}([s,\infty))$, and $t = \sup\{s>0:\mathrm{Vol}_M(K_s)>0\}$. Although K_s and t clearly depend on f (and consequently on μ), this fact is not reflected in the notation — in the sequel it shall be clear from the context which f and μ are being referred to.

1.2. **Proposition.** *Let* $\mu\in\mathcal{P}_M$ *be absolutely continuous w.r.t.* Vol_M, *as above. Then*

$$\mu(A) = \int_0^t \lambda_{K_s}(A)\ \mathrm{Vol}_M(K_s)\ ds \ , \qquad A\in\mathcal{B}_M \ .$$

Proof. Note that μ being a probability ensures that $\mathrm{Vol}_M(K_s)<\infty$ for all $s>0$.

We can then write, for $A\in\mathcal{B}_M$,

$$\mu(A) = \int_A f(x)\,dx = \int_{\mathbb{R}^M} \mathbb{1}_A(x) \int_0^{f(x)} 1\,ds\,dx = \int_{\mathbb{R}^M} \mathbb{1}_A(x) \int_0^{\infty} \mathbb{1}_{K_s}(x)\,ds\,dx$$

$$= \int_0^{\infty} \int_{\mathbb{R}^M} \mathbb{1}_A(x)\,\mathbb{1}_{K_s}(x)\,dx\,ds = \int_0^{\infty} \mathrm{Vol}_M(A \cap K_s)\,ds$$

$$= \int_0^{t} \lambda_{K_s}(A)\ \mathrm{Vol}_M(K_s)\ ds\ ,$$

due to the measurability of $(x,s) \longmapsto \mathbb{1}_A(x)\,\mathbb{1}_{K_s}(x)$ and the easy fact that $s_1>s_2>0 \Rightarrow K_{s_1} \subset K_{s_2}$.

1.3. **Definition.** *Let* $\mu \in \mathcal{P}_M$ *be absolutely continuous w.r.t.* Vol_M. μ *is said to be* unimodal *(in symbols,* $\mu \in \mathcal{U}_M$*) if there is a version* f *of* $d\mu/d\mathrm{Vol}_M$ *which is an even function and for which* K_s *are convex for all* $s \in (0,t)$.

1.4. **Definition.** *Given* $\mu,\nu \in \mathcal{P}_M$, μ *is said to be* more peaked than ν *(in symbols,* $\mu \succ \nu$*) if* $\mu(C) \geq \nu(C)$ *for all closed, convex, symmetric subsets* C *of* $\mathbb{R}^M$ *(the symmetry of* C *meaning that* $-C=C$*).*

1.5. **Lemma.** *Let* $\mu,\nu \in \mathcal{P}_M$ *be such that* $\mu \succ \nu$. *Then*

$$\int_{\mathbb{R}^M} g\,d\mu \geq \int_{\mathbb{R}^M} g\,d\nu \tag{1.1}$$

for all non-negative real functions g *on* $\mathbb{R}^M$ *that are bounded, even, upper semicontinuous and log-concave (the* log-concavity *of* g *meaning that* $\{x \in \mathbb{R}^M : g(x)>0\}$ *is convex and* log g *is concave on this set).*

Proof. Let g be a function such as above and set $G = \sup_{x \in \mathbb{R}^M} g(x)$.
Define

$$g_n = \sum_{k=1}^{n} \frac{G}{n}\, \mathbb{1}_{g^{-1}([Gk/n,\infty))}\ , \qquad n \in \mathbb{N}\ .$$

Note that $g^{-1}([Gk/n,\infty))$ is closed (by the upper semicontinuity of g), symmetric (because g is even) and convex (this following from the log-concavity of g).

The fact that $\mu \succ \nu$ implies then that

$$\begin{aligned}\int_{\mathbb{R}^M} g_n(x)\, d\mu(x) &= \sum_{k=1}^{n} \frac{G}{n}\, \mu(g^{-1}([Gk/n,\infty))) \\ &\geq \sum_{k=1}^{n} \frac{G}{n}\, \nu(g^{-1}([Gk/n,\infty))) = \int_{\mathbb{R}^M} g_n(x)\, d\nu(x) \ ,\end{aligned} \tag{1.2}$$

so that (1.1) holds for g_n, $n \in \mathbb{N}$.

Given x and n we have $\frac{Gk}{n} \leq g(x) < \frac{G(k+1)}{n}$ for some $k \in \{0,\dots,n\}$. Since this k will vary with n, we shall denote it by k_n to show the dependence on n. We can then write the previous inequalities as $\frac{Gk_n}{n} \leq g(x) < \frac{G(k_n+1)}{n}$. Note now that

$$g_n(x) = \sum_{k=1}^{k_n} \frac{G}{n} = \frac{Gk_n}{n} \leq g(x) < \frac{Gk_n}{n} + \frac{G}{n} = g_n(x) + \frac{G}{n}$$

for $n > n_0$ say, except when $g(x)=0$, in which case $g_n(x)=0$ for all $n \in \mathbb{N}$. We have then that, if $g(x)=0$, $\lim_n g_n(x) = g(x)$; if $g(x)>0$, $|g(x)-g_n(x)| < \frac{G}{n}$ and therefore also $\lim_n g_n(x) = g(x)$. The bounded convergence theorem thus applies, together with (1.2), to yield

$$\int_{\mathbb{R}^M} g(x)\, d\mu(x) = \lim_n \int_{\mathbb{R}^M} g_n(x)\, d\mu(x) \geq \lim_n \int_{\mathbb{R}^M} g_n(x)\, d\nu(x) = \int_{\mathbb{R}^M} g(x)\, d\nu(x) \ ,$$

as required.

1.6. **Definition.** $\mu \in \mathcal{P}_M$ *is said to be* log-concave *if for all* $\vartheta \in (0,1)$ *and all* $A,B \in \mathcal{B}_M$ *such that* $\vartheta A + (1-\vartheta)B \in \mathcal{B}_M$ *the following holds:* $\mu(\vartheta A + (1-\vartheta)B) \geq \mu(A)^{\vartheta}\mu(B)^{1-\vartheta}$.

1.7. **Lemma** (see [Rinott 1976] for a proof). *Let* $\mu \in \mathcal{P}_M$ *be absolutely continuous w.r.t.* Vol_M. *Then* μ *is log-concave* if and only if *there is a log-concave version of* $d\mu/d\mathrm{Vol}_M$.

1.8. **Proposition.** *Let* $\mu \in U_M$ *and* $\mu_1,\mu_2 \in \mathcal{P}_N$ *with* $\mu_1 \succ \mu_2$. *Then* $\mu\times\mu_1 \succ \mu\times\mu_2$.

Proof. The notation in this proof is consistent with Definition 1.3.

Furthermore, A will denote an arbitrary closed, convex, symmetric subset A of $\mathbb{R}^M \times \mathbb{R}^N$.

Let us first prove that $\lambda_{K_s} \times \mu_1 \succ \lambda_{K_s} \times \mu_2$, $s \in (0,t)$.

The parameter s being fixed now, we shall simply write K instead of K_s.

We have

$$\begin{aligned}(\lambda_K \times \mu_1)(A) &= \int_{\mathbb{R}^M \times \mathbb{R}^N} \mathbb{1}_A(x,y)\ d(\lambda_K \times \mu_1)(x,y) \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^M} \mathbb{1}_A(x,y)\ d\lambda_K(x)\ d\mu_1(y)\ .\end{aligned} \tag{1.3}$$

Denote $g(y) := \int_{\mathbb{R}^M} \mathbb{1}_A(x,y)\, d\lambda_K(x)$, $y \in \mathbb{R}^N$. From the definition of K and the unimodality of μ it follows that the non-negative function g is bounded (it is even ≤ 1) and even. It is also upper semicontinuous: in fact, all we have to prove is that, given any sequence $y_n \xrightarrow[n]{} y$, $\limsup_n g(y_n) \le g(y)$. Consider then $(y_n)_n$ such that $y_n \xrightarrow[n]{} y$. It is trivial that the functions $\mathbb{1}_A(.,y_n)$, $n \in \mathbb{N}$, and 1 are λ_K-integrable; $\mathbb{1}_A(x,y_n) \le 1$, $x \in \mathbb{R}^M$, $n \in \mathbb{N}$; and $\limsup_n \int_{\mathbb{R}^M} \mathbb{1}_A(x,y_n)\ d\lambda_K(x) \ge 0 > -\infty$, so that Fatou's lemma applies to yield

$$\limsup_n g(y_n) \le \int_{\mathbb{R}^M} \limsup_n \mathbb{1}_A(x,y_n)\ d\lambda_K(x)\ . \tag{1.4}$$

Note now that, given $x \in \mathbb{R}^M$, $\mathbb{1}_A(x,y_n) \le \mathbb{1}_A(x,y)$ for $n > n_0$ say (indeed, this is trivially true if $(x,y) \in A$; if $(x,y) \notin A$, since A is closed there is $\varepsilon > 0$ such that $B^M_\infty(x,\varepsilon) \times B^N_\infty(y,\varepsilon) \subset {}^cA$, so that for $n > n_0$, say, $y_n \in B^N_\infty(y,\varepsilon)$ and consequently $(x,y_n) \notin A$), and therefore $\limsup_n \mathbb{1}_A(x,y_n) \le \mathbb{1}_A(x,y)$. This and (1.4) gives $\limsup_n g(y_n) \le g(y)$, thus finishing the proof of the upper semicontinuity of g.

In order to be able to apply Lemma 1.5 we still need to show that g is log-concave.

Recall that $g(y) = \int_{\mathbb{R}^M} \mathbb{1}_A(x,y)\, d\lambda_K(x) = \lambda_K(A^y)$, where $A^y := \{x \in \mathbb{R}^M : (x,y) \in A\}$, $y \in \mathbb{R}^N$, and that, by Lemma 1.7, λ_K is a log-concave measure (in fact, the convexity of K yields the log-concavity of $\frac{1}{\mathrm{Vol}_M(K)} \mathbb{1}_K$).

Given then $y_1,y_2\in\mathbb{R}^N$ and $\vartheta\in(0,1)$, we can write, since the convexity of A ensures that $\vartheta A^{y_1}+(1-\vartheta)A^{y_2}\subset A^{\vartheta y_1+(1-\vartheta)y_2}$,

$$g(\vartheta y_1+(1-\vartheta)y_2) = \lambda_{\mathbf{K}}(A^{\vartheta y_1+(1-\vartheta)y_2}) \geq \lambda_{\mathbf{K}}(\vartheta A^{y_1}+(1-\vartheta)A^{y_2})$$

$$\geq \lambda_{\mathbf{K}}(A^{y})^{\vartheta}\lambda_{\mathbf{K}}(A^{y_2})^{1-\vartheta} = g(y_1)^{\vartheta} g(y_2)^{1-\vartheta},$$

which proves the log-concavity of g.

We can thus apply Lemma 1.5 to obtain, from (1.3) and the hypothesis $\mu_1 \succ \mu_2$,

$$(\lambda_{\mathbf{K}_s}\times\mu_1)(A) \geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^M} \mathbb{1}_{\mathbf{A}}(x,y)\ d\lambda_{\mathbf{K}_s}(x)\, d\mu_2(y)$$

$$= \int_{\mathbb{R}^M\times\mathbb{R}^N} \mathbb{1}_{\mathbf{A}}(x,y)\ d(\lambda_{\mathbf{K}_s}\times\mu_2)(x,y) = (\lambda_{\mathbf{K}_s}\times\mu_2)(A)$$

for all $s\in(0,t)$.

Now we use this and Proposition 1.2 to obtain

$$(\mu\times\mu_1)(A) = \int_{\mathbb{R}^M\times\mathbb{R}^N} \mathbb{1}_{\mathbf{A}}(x,y)\ d\mu(x)\, d\mu_1(y) = \int_{\mathbb{R}^N} \mu(A^y)\, d\mu_1(y)$$

$$= \int_{\mathbb{R}^N}\int_0^t \lambda_{\mathbf{K}_s}(A^y)\,\mathrm{Vol}_{\mathbf{M}}(K_s)\, ds\ \ d\mu_1(y)$$

$$= \int_0^t\int_{\mathbb{R}^N} \lambda_{\mathbf{K}_s}(A^y)\ d\mu_1(y)\,\mathrm{Vol}_{\mathbf{M}}(K_s)\, ds$$

$$= \int_0^t (\lambda_{\mathbf{K}_s}\times\mu_1)(A)\,\mathrm{Vol}_{\mathbf{M}}(K_s)\, ds$$

$$\geq \int_0^t (\lambda_{\mathbf{K}_s}\times\mu_2)(A)\,\mathrm{Vol}_{\mathbf{M}}(K_s)\, ds = (\mu\times\mu_2)(A),$$

taking into account the fact that $(s,y) \longmapsto \lambda_{\mathbf{K}_s}(A^y)\,\mathrm{Vol}_{\mathbf{M}}(K_s) = \int_{\mathbb{R}^M} \mathbb{1}_{\mathbf{A}}(x,y)\,\mathbb{1}_{\mathbf{K}_s}(x)\, dx$ is a measurable function (this can be seen by showing, much as in the case of g above, that this function is in fact upper semicontinuous).

1.9. **Corollary.** *Suppose* $\mu_1,\mu_2\in\mathcal{P}_{\mathbf{N}}$, $\nu_1,\nu_2\in\mathcal{P}_{\mathbf{M}}$, *with* $\mu_1\succ\mu_2$, $\nu_1\succ\nu_2$, $\mu_1\in\mathrm{U}_{\mathbf{N}}$, $\nu_2\in\mathrm{U}_{\mathbf{M}}$. *Then* $\mu_1\times\nu_1 \succ \mu_2\times\nu_2$.

Proof. We apply Proposition 1.8 to obtain $\mu_1\times\nu_1 \succ \mu_1\times\nu_2$ and $\nu_2\times\mu_1 \succ \nu_2\times\mu_2$, or $\mu_1\times\nu_2 \succ \mu_2\times\nu_2$, of course, from which follows $\mu_1\times\nu_1 \succ \mu_2\times\nu_2$, as required.

2. Quasi-Banach spaces, operators and geometric quantities

2.1. In this section the scalar field can either be real or complex.

We begin by mentioning some facts about quasi-Banach spaces in a rather informal way, for most of them can be derived by easy adaptation of the corresponding proofs used in the Banach situation (compare for example with [Taylor/Lay 1980]).

We recall first that the only difference between the formal definitions of a Banach space and a *quasi-Banach space* X is that in the latter one instead of a norm we have a *quasi-norm*, i.e., a function $\|\cdot\|$ with the same defining properties of a norm with the exception of the triangle inequality, which is replaced by the following *quasi-triangle inequality*:

$$\|x+y\| \le C\ (\|x\|+\|y\|) \quad \text{for all } x,y\in X\ , \tag{2.1}$$

where $C\ge 1$ is independent of x and y.

The *topology in* X is defined by the *basis of* (not necessarily open) *neighbourhoods* constituted by the sets $\{y\in X:\ \|y-x\|<1/n\}$, $x\in X$, $n\in\mathbb{N}$, thus endowing X with a topological vector space structure (cf. [Köthe 1969, p. 160]).

It is well-known (see, for example, [König 1986, p. 47]) that a quasi-norm $\|\cdot\|$ on X is always equivalent to a t-norm $|||\cdot|||$ on X for $t\in(0,1]$ such that $C=2^{1/t-1}$. Here *equivalent* means that there are $c_1,c_2>0$ such that

$$c_1\|x\| \le |||x||| \le c_2\|x\| \quad \text{for all } x\in X \tag{2.2}$$

(or, what is the same, that $\|\cdot\|$ and $|||\cdot|||$ define the same topology in X), and a

t-*norm* $|||\cdot|||$ has all the defining properties of a norm except again the triangle inequality, this being replaced by the t-*triangle inequality*:

$$|||x+y|||^t \le |||x|||^t + |||y|||^t \quad \text{for all } x,y\in X \tag{2.3}$$

(of course, a t-norm is also a quasi-norm).

A linear map T between quasi-Banach spaces (over the same scalar field) is an *operator* (that is, is continuous) if and only if it is *bounded*, that is, iff there is a constant M such that $\|Tx\|\le M\|x\|$ for every x in the domain $\mathcal{D}(T)$ of T (however, only in the context of Hilbert spaces, the name operator will also be given to an unbounded linear map — see 4.8). In this case

$$\|T\| := \sup_{\|x\|\le 1}\|Tx\| \quad \left(= \sup_{x\ne 0}\frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1}\|Tx\| \ \text{ if } \mathcal{D}(T)\ne\{0\}\right) \tag{2.4}$$

is finite and for fixed quasi-Banach spaces X and Y the map $\|\cdot\|: \mathcal{L}(X,Y) \longrightarrow \mathbb{R}$ defined by (2.4) is a quasi-norm in $\mathcal{L}(X,Y)$, that is, in the linear space of all operators from X into Y. We can even say that the constant in the quasi-triangle inequality for this *operator quasi-norm* can be taken to be that for the quasi-norm in Y; moreover, if the latter is a t-norm, the former is also a t-norm for the same $t\in(0,1]$.

As a natural extension of the definition of s-function in the context of Banach spaces, we introduced in [Caetano a] the following

2.2. DEFINITION. *An* s-function *is a map* $\mathbf{s}$ *assigning to each operator* T *between quasi-Banach spaces a sequence* $(s_k(T))_k$ *such that*

(i) $\|T\| = s_1(T) \ge s_2(T) \ge \dots \ge 0$;

(ii) $(s_{k+m-1}(S+T))^t \le (s_k(S))^t + (s_m(T))^t$ *for all* $k,m\in\mathbb{N}$, *all* $S,T\in\mathcal{L}(X,Y)$, *all quasi-Banach spaces* X *and t-Banach spaces* Y, *for all* $t\in(0,1]$;

(iii) $s_k(RTS) \le \|R\|\, s_k(T)\, \|S\|$ *for all* $k\in\mathbb{N}$, *all* $S\in\mathcal{L}(X_0,X)$, $T\in\mathcal{L}(X,Y)$, $R\in\mathcal{L}(Y,Y_0)$ *and all quasi-Banach spaces* X_0, X, Y, Y_0 ;

(iv) $\dim T < k \Rightarrow s_k(T) = 0$;

(v) $s_M(I_2^2(M)) = 1$, *where* $I_2^2(M) : l_2^M \longrightarrow l_2^M$ *is the identity map.*

As a straightforward consequence of (ii) of the previous definition we have

2.3. **PROPOSITION.** *If* **s** *is an s-function and* X *and* Y *are, respectively, a quasi-Banach space and a t-Banach space,* $t \in (0,1]$, *then, with* $T_j : X \longrightarrow Y$ *operators,* $r_j, N \in \mathbb{N}$, $j=1,\dots,N$, *and* $\sigma := \sum_{j=1}^{N} r_j$, *we have*

$$\left(s_{\sigma-(N-1)}\Big(\sum_{j=1}^{N} T_j\Big) \right)^t \le \sum_{j=1}^{N} \left(s_{r_j}(T_j) \right)^t .$$

2.4. **PROPOSITION.** *If* **s** *is an s-function and* $0<p\le 2$ *then*

$$s_k(I_2^p(M)) \ge k^{1/2-1/p} \text{ for all } k,M \in \mathbb{N} \text{ with } k \le M,$$

where $I_2^p(M) : l_p^M \longrightarrow l_2^M$ *is the natural embedding.*

Proof. Using 2.2(iii),(v) we can write $1 = s_k(I_2^2(k)) \le s_k(I_2^p(k)) \, \|I_p^2(k)\| \le s_k(I_2^p(M)) \, k^{1/p-1/2}$, hence the result.

2.5. Examples of s-functions:

2.5.1. $(a_k(\cdot))_k$, with $a_k(T) := \inf \{ \|T-S\| : S \in \mathcal{L}(X,Y), \dim S < k \}$ the *approximation numbers* of $T \in \mathcal{L}(X,Y)$;

2.5.2. $(c_k(\cdot))_k$, with $c_k(T) := \inf \{ \|T|_M\| : M \lhd X, \operatorname{codim} M < k \}$ the *Gelfand numbers* of $T \in \mathcal{L}(X,Y)$, where $\lhd$ means 'is a closed subspace of';

2.5.3. $(x_k(\cdot))_k$, with $x_k(T) := \sup \{a_k(TA) : A \in \mathcal{L}(l_2,X), \|A\| \le 1\}$ the *Weyl numbers* of $T \in \mathcal{L}(X,Y)$.

As to proofs of these facts, they are an immediate adaptation of the corresponding ones for the Banach situation.

2.6. **PROPOSITION.** *Given a quasi-Banach space* X, *a t-Banach space* Y *for some* $t \in (0,1]$, *and* $T \in \mathcal{L}(X,Y)$,

(a) *for every s-function we have* $s_k(T) \le a_k(T)$, $k \in \mathbb{N}$;

(b) *if* X *is a Hilbert space then* $c_k(T) = a_k(T)$, $k \in \mathbb{N}$;

(c) $x_k(T) = \sup \{ c_k(TA) : A \in \mathcal{L}(l_2, X),\ \|A\| \le 1 \}$.

Proof. (a) and (c) are easy. As to (b), compare with [König 1986, p. 30].

In Chapter II we will need the following property of the Weyl numbers (called *multiplicativity*), which is an improvement of 2.2(iii):

2.7. **Proposition.** *Given* $k,m \in \mathbb{N}$, *quasi-Banach spaces* X, Y, Z *and operators* $S \in \mathcal{L}(X,Y)$, $T \in \mathcal{L}(Y,Z)$, *the following holds:*

$$x_{k+m-1}(TS) \le x_k(T)\, x_m(S) .$$

Proof. This is essentially the proof of [Pietsch 1987, p. 94].

Given $\varepsilon > 0$ and $A \in \mathcal{L}(l_2, X)$ with $\|A\| \le 1$, choose $B \in \mathcal{L}(l_2, Y)$ with $\dim B < m$ such that $\|SA - B\| \le a_m(SA) + \varepsilon$, and $C \in \mathcal{L}(l_2, Z)$ with $\dim C < k$ such that $\|T(SA-B) - C\| \le a_k(T(SA-B)) + \varepsilon$. These inequalities and the fact that $\dim(C+TB) \le \dim C + \dim B < k+m-1$ imply that

$$a_{k+m-1}(TSA) \le \|TSA - C - TB\| \le a_k(T(SA-B)) + \varepsilon \le x_k(T)\, \|SA-B\| + \varepsilon$$

$$\le x_k(T)\, a_m(SA) + x_k(T)\, \varepsilon + \varepsilon \le x_k(T)\, x_m(S) + (1 + x_k(T))\, \varepsilon ,$$

from which follows $x_{k+m-1}(TS) \le x_k(T)\, x_m(S) + (1 + x_k(T))\, \varepsilon$. Letting $\varepsilon \to 0^+$, we obtain the desired result.

2.8. There is another geometric quantity which will be of interest to us, but we need some preliminary considerations before we can present it.

Given a subspace H of $\mathbb{K}^M$ (this can be either $\mathbb{R}^M$ or $\mathbb{C}^M$, which we shall assume endowed with the usual Euclidean norms wherever necessary), $M \in \mathbb{N}$, we define a measure in H in the following way. Consider in H the Hilbert structure induced by the usual inner product in $\mathbb{K}^M$, choose an orthonormal basis $\{z_1, \dots, z_k\}$ of H (with $k = \dim H$) and complete it orthonormally by means of $z_{k+1}, \dots, z_M$ to get also an orthonormal basis of $\mathbb{K}^M$ (use the Gram-Schmidt process whenever necessary). Consider the endomorphism R of $\mathbb{K}^M$ that takes $\delta_j^M := (\delta_{jm})_m$ (where δ_{jm} is the Kronecker symbol) to z_j, $j=1, \dots, M$, let J_k: $\mathbb{K}^k \longrightarrow \mathbb{K}^M$ be the natural embedding and R_k: $\mathbb{K}^k \longrightarrow H$ be given by $R_k x = RJ_k x$, $x \in \mathbb{K}^k$. Induce then a measure structure in H by means of R_k and Vol_k (this is Lebesgue measure in $\mathbb{R}^k$ if $\mathbb{K}=\mathbb{R}$ and the measure in $\mathbb{C}^k$ induced in the usual way from Lebesgue measure in $\mathbb{R}^{2k}$ if $\mathbb{K}=\mathbb{C}$) and denote the corresponding (Lebesgue) measure in H by Vol_H.

Whenever it seems necessary, Vol_H will also denote the corresponding Lebesgue-Borel measure in H, or the corresponding outer measure in H.

The (outer) measure just defined in H is independent of the particular orthonormal bases $\{z_j\}_j$ considered in the definition. Let us dwell a bit upon this for the case $\mathbb{K}=\mathbb{C}$. Besides the R_k above, consider another such map, R'_k say, constructed in the same way from another basis of $\mathbb{C}^M$ – $\{w_1,\dots,w_M\}$ say, such that $\{w_1,\dots,w_k\}$ is a basis of H. In view of some results to be deduced below, we do not make any assumption of orthonormality for this basis at the moment. Denoting by Vol'_H the corresponding outer measure defined in H, and by $I_k : \mathbb{R}^{2k} \longrightarrow \mathbb{C}^k$ the natural identification, we have

$$\begin{aligned} \mathrm{Vol}'_H(A) &= \mathrm{Vol}_k(R_k'^{-1}A) = \mathrm{Vol}_{2k}(I_k^{-1}R_k'^{-1}A) \\ &= \mathrm{Vol}_{2k}((R'_k I_k)^{-1}(R_k I_k)(R_k I_k)^{-1}A) \qquad (2.5) \\ &= |\det((R'_k I_k)^{-1}(R_k I_k))| \, \mathrm{Vol}_H(A) \qquad \text{for } A \subset H . \end{aligned}$$

Denote $T := (R'_k I_k)^{-1}(R_k I_k)$ and assume now that $\{w_j\}_j$ is indeed orthonormal. Note that, for $j=1,\dots,k$,

$$R'_k I_k \delta_{2j}^{2k} = R'_k i\delta_j^k = iR'_k \delta_j^k = iw_j ,$$

$$R'_k I_k \delta_{2j-1}^{2k} = R'_k \delta_j^k = w_j ,$$

and analogously with R'_k and w_j replaced respectively by R_k and z_j. This easily implies that $\|R_kI_kx\|_{\mathbb{C}^M} = \|x\|_{\mathbb{R}^{2k}} = \|R'_kI_kx\|_{\mathbb{C}^M}$ for all $x\in\mathbb{R}^{2k}$, and, consequently, $\|Tx\|_{\mathbb{R}^{2k}} = \|x\|_{\mathbb{R}^{2k}}$ for all $x\in\mathbb{R}^{2k}$. Hence (cf. [Taylor/Lay 1980, pp. 89-90]), for all $x,y\in\mathbb{R}^{2k}$, $(Tx,Ty)_{\mathbb{R}^{2k}} = (x,y)_{\mathbb{R}^{2k}}$, that is, $T^*T = Id_{\mathbb{R}^{2k}}$, and therefore $|\det T| = 1$. This and (2.5) prove that Vol_H and Vol'_H coincide. The case $\mathbb{K}=\mathbb{R}$ can be dealt with similarly.

2.9. LEMMA. *Let* S *be an endomorphism of* $\mathbb{C}^k$, *and* Vol_k *the outer measure in* $\mathbb{C}^k$ *induced in the usual way from Lebesgue outer measure in* $\mathbb{R}^{2k}$. *Then*

$$Vol_k(SA) = |\det S|^2 Vol_k(A), \qquad A\subset\mathbb{C}^k.$$

Proof. From (2.5), with M=k, $H=\mathbb{C}^k$ and $R_k=Id_{\mathbb{C}^k}$, we obtain $Vol'_{\mathbb{C}^k}(A) = |\det((R'_kI_k)^{-1}I_k)|\, Vol_k(A)$ for all $A\subset\mathbb{C}^k$, so that

$$\frac{Vol_k(SA)}{Vol_k(A)} = \frac{Vol'_{\mathbb{C}^k}(SA)}{Vol'_{\mathbb{C}^k}(A)}, \qquad A\subset\mathbb{C}^k, \tag{2.6}$$

no matter what R'_k is considered in accordance with the hypotheses that yielded (2.5).

Let then R'_k be defined by means of a basis $\{w_j\}_j$ with respect to which the matrix of S has the Jordan canonical form, and set $T:=(R'_kI_k)^{-1}S(R'_kI_k)$. Since the determinant of the matrix of T with respect to the canonical basis of $\mathbb{R}^{2k}$ is equal to $|\det S|^2$ (see [Caetano 1987, pp. 70-71] or [Carl/Triebel 1980, pp. 129-130]), then $\det T = |\det S|^2$ and therefore

$$\frac{Vol'_{\mathbb{C}^k}(SA)}{Vol'_{\mathbb{C}^k}(A)} = \frac{Vol_{2k}((R'_kI_k)^{-1}SA)}{Vol_{2k}((R'_kI_k)^{-1}A)} = \frac{Vol_{2k}(T(R'_kI_k)^{-1}A)}{Vol_{2k}((R'_kI_k)^{-1}A)}$$
$$= |\det T| = |\det S|^2.$$

Hence, by (2.6), the required result follows.

2.10. DEFINITION. *Let* X *be a quasi-Banach space and* $T\in\mathcal{L}(X,l_2^M)$ *for* *some* $M\in\mathbb{N}$. *The* volume numbers *of* T *are defined by*

$$v_k(T) = \sup\left\{\left(\frac{\mathrm{Vol}_H(H\cap TB_X)}{\mathrm{Vol}_k(B_2^k)}\right)^{1/k'} : H\lhd l_2^M,\ \dim H = k\right\}, \quad k=1,\dots,M,$$

where $k'=k$ if $\mathbb{K}=\mathbb{R}$, $k'=2k$ if $\mathbb{K}=\mathbb{C}$, $B_X:=\{x\in X: \|x\|_X\le 1\}$ *and* $B_2^k:=B_{l_2^k}$.

We note that these numbers, considered for the first time in [Caetano c], are not the *volume ratio* numbers that appear in the literature (see, for example, [Pajor/Tomczak-Jaegermann 1989]), although in the above definition we also use a ratio between volumes. There is indeed at least a formal duality between the volume ratio numbers and the volume numbers.

2.11. It is easy to see that the volume numbers couldn't possibly give rise to an s-function even if we had managed to define them for all operators between quasi-Banach spaces. Indeed, consider the diagonal operator $D: l_2^M \longrightarrow l_2^M$ given by $D(\xi_j)_{j=1}^M = (\sigma_j\xi_j)_{j=1}^M$, where $\sigma_1\ge\dots\ge\sigma_M\ge 0$ are fixed. It is well-known (see for example [Pietsch 1980, 11.3.2]) that in this case we have, for any s-function $\mathbf{s}$, $s_M(D)=\sigma_M$, while

$$v_M(D) = \left(\frac{\mathrm{Vol}_M(DB_2^M)}{\mathrm{Vol}_M(B_2^M)}\right)^{1/M'} = |\det D|^{1/M} = \left(\prod_{j=1}^M \sigma_j\right)^{1/M}.$$

To be more precise, if $M=2$, $\sigma_1=1$ and $\sigma_2=1/2$ then $s_2(D) = 1/2 \ne 1/\sqrt{2} = v_2(D)$.

2.12. **Proposition.** *Adding the symbol* $\mathbb{R}$ *to the ambiguous notation* iff *real spaces are meant, the following holds:*

$$v_k(I_2^p(M)) \le 2^{1/\bar{p}-1/2}\, v_{2k}(I_2^p(2M;\mathbb{R})), \quad k\in\{1,\dots,M\},\ M\in\mathbb{N}, \tag{2.7}$$

where $\bar{p} = \min\{p,2\}$ *and* $p\in(0,\infty]$.

Proof. Let $I_k:\mathbb{R}^{2k}\longrightarrow\mathbb{C}^k$ and $I_M:\mathbb{R}^{2M}\longrightarrow\mathbb{C}^M$ be the natural identifications. We have

$$\mathrm{Vol}_k(B_2^k) = \mathrm{Vol}_{2k}(I_k^{-1}B_2^k) = \mathrm{Vol}_{2k}(B_2^{2k}(\mathbb{R})) \tag{2.8}$$

and, using the notation of 2.8 (in particular H is a k-dimensional subspace of $\mathbb{C}^M$),

$$\begin{aligned}
\mathrm{Vol}_H(H\cap B_p^M) &= \mathrm{Vol}_k(R_k^{-1}(H\cap B_p^M)) = \mathrm{Vol}_k((RJ_k)^{-1}H\cap(RJ_k)^{-1}B_p^M) \\
&= \mathrm{Vol}_{2k}((RJ_kI_k)^{-1}H\cap(RJ_kI_k)^{-1}B_p^M) \\
&= \mathrm{Vol}_{2k}((RJ_kI_k)^{-1}I_MI_M^{-1}H\cap (RJ_kI_k)^{-1}I_MI_M^{-1}B_p^M) \qquad (2.9)\\
&\le \mathrm{Vol}_{2k}((I_M^{-1}RJ_kI_k)^{-1}I_M^{-1}H\cap (I_M^{-1}RJ_kI_k)^{-1}2^{1/\bar{p}-1/2}B_p^{2M}(\mathbb{R})) \\
&= 2^{(1/\bar{p}-1/2)2k}\,\mathrm{Vol}_{2k}((I_M^{-1}RJ_kI_k)^{-1}(I_M^{-1}H\cap B_p^{2M}(\mathbb{R}))) \\
&= 2^{(1/\bar{p}-1/2)2k}\,\mathrm{Vol}_{I_M^{-1}H}(I_M^{-1}H\cap B_p^{2M}(\mathbb{R}))\,,
\end{aligned}$$

the last equality coming from the fact that $I_M^{-1}RJ_kI_k = R'J_k'$, with $R' := I_M^{-1}RI_M$ an orthogonal transformation in $\mathbb{R}^{2M}$, $J_k' := I_M^{-1}J_kI_k : \mathbb{R}^{2k}\longrightarrow\mathbb{R}^{2M}$ the natural embedding and the range of $R'J_k'$ equal to the 2k-dimensional subspace $I_M^{-1}H$ of $\mathbb{R}^{2M}$. Hence, from (2.8) and (2.9),

$$\begin{aligned}
v_k(I_2^p(M)) &= \sup\left\{\left(\frac{\mathrm{Vol}_H(H\cap B_p^M)}{\mathrm{Vol}_k(B_2^k)}\right)^{\frac{1}{2k}} : H\lhd l_2^M,\ \dim H=k\right\} \\
&\le \sup\left\{\left(\frac{2^{(1/\bar{p}-1/2)2k}\,\mathrm{Vol}_{I_M^{-1}H}(I_M^{-1}H\cap B_p^{2M}(\mathbb{R}))}{\mathrm{Vol}_{2k}(B_2^{2k}(\mathbb{R}))}\right)^{\frac{1}{2k}} : H\lhd l_2^M,\ \dim H=k\right\} \\
&\le 2^{1/\bar{p}-1/2}\, v_{2k}(I_2^p(2M;\mathbb{R}))\,,
\end{aligned}$$

as required.

3. The spaces B^s_{pq} and F^s_{pq}

Throughout this section Ω will stand for a non-empty open bounded C^∞-domain in the n-dimensional Euclidean space $\mathbb{R}^n$ (see [Triebel 1983, 3.2.1]); $s,s_1,s_2\in\mathbb{R}$, $p,p_1,p_2,q,q_1,q_2\in(0,\infty]$; and, unless otherwise stated, all function spaces under consideration will be complex ones.

3.1. Definition. (i) $L_p(\mathbb{R}^n)$ *is the usual quasi-Banach space of* p*-integrable (measurable, essentially bounded in the case* $p=\infty$*) functions on* $\mathbb{R}^n$ *with respect to Lebesgue measure; we denote its quasi-norm by* $\|\cdot\|_p$.

(ii) $S(\mathbb{R}^n)$ *is the space of rapidly decreasing infinitely differentiable functions on* $\mathbb{R}^n$ *equipped with the usual topology — see [Triebel 1983, 1.2.1].*

(iii) $S'(\mathbb{R}^n)$ *is the topological dual of* $S(\mathbb{R}^n)$ *— that is, the space of tempered distributions — equipped with the strong topology.*

(iv) $D'(\Omega)$ *is the usual space of distributions on* Ω.

(v) $L_p^Q(\mathbb{R}^n) = \{f\in S'(\mathbb{R}^n)\colon \operatorname{supp} f^\wedge \subset Q,\ \|f\|_p<\infty\}$ for Q compact in $\mathbb{R}^n$ — *see also 3.2 below.*

3.2. If $\varphi\in S(\mathbb{R}^n)$ is such that

$$\operatorname{supp}\varphi \subset 2\overset{o}{B}{}_2^n \quad \text{and} \quad \varphi(x)=1 \text{ if } x\in B_2^n \tag{3.1}$$

then we define $\varphi_j(x) := \varphi(2^{-j}x)-\varphi(2^{-j+1}x)$ for each $j\in\mathbb{N}$, and $\varphi_o:=\varphi$. Note that $\sum_{j=o}^{\infty}\varphi_j(x) = 1$ for all $x\in\mathbb{R}^n$ and thus the φ_j form a (dyadic) resolution of unity. If $f\in S'(\mathbb{R}^n)$ we denote its Fourier and inverse Fourier transform respectively by $f^\wedge$ and $f^\vee$. Recall that by a Paley-Wiener-Schwartz theorem — see [Triebel 1983, 1.2.1] — we have for the φ_j, $j\in\mathbb{N}_o$, and f considered above that $(\varphi_j f^\wedge)^\vee$ is an entire analytic function on $\mathbb{R}^n$.

3.3. Definition. (i) $B^s_{pq}(\mathbb{R}^n)$ *is the collection of all* $f\in S'(\mathbb{R}^n)$ *such that*

$$\|f|B^s_{pq}(\mathbb{R}^n)\| := \Big(\sum_{j=o}^{\infty} 2^{jsq}\,\|(\varphi_j f^\wedge)^\vee\|_p^q\Big)^{1/q} \tag{3.2}$$

(with the usual modification if $q=\infty$*) is finite.*

(ii) *If* $p\neq\infty$, $F^s_{pq}(\mathbb{R}^n)$ *is the collection of all* $f\in S'(\mathbb{R}^n)$ *such that*

$$\|f|F^s_{pq}(\mathbb{R}^n)\| := \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq}\,|(\varphi_j f^\wedge)^\vee|^q\Big)^{1/q}\Big\|_p \tag{3.3}$$

(with the usual modification if $q=\infty$*) is finite.*

3.4. *Remark.* (i) $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are quasi-Banach spaces (Banach spaces if $p,q\geq1$) with the quasi-norms given by (3.2) and (3.3) respectively. They are independent of the function $\varphi\in S(\mathbb{R}^n)$ chosen according to (3.1) in the sense of equivalent quasi-norms, and this justifies the omission of the function φ in consideration on the left-hand side of (3.2) and (3.3).

(ii) Also in the sense of equivalent quasi-norms, $F^s_{p2}(\mathbb{R}^n)$, $s\in\mathbb{N}_o$, $1<p<\infty$, are the more familiar Sobolev spaces $W^s_p(\mathbb{R}^n)$, and $B^s_{pq}(\mathbb{R}^n)$, $s>0$, $1\leq p<\infty$, $1\leq q\leq\infty$, are the Besov spaces $\Lambda^s_{pq}(\mathbb{R}^n)$.

(iii) For more information about these spaces and their connection with the classical ones, we refer to the book of Triebel [1983].

3.5. DEFINITION. (i) $B^s_{pq}(\Omega)$ *is the collection of all* $f\in D'(\Omega)$ *such that there exists a* $g\in B^s_{pq}(\mathbb{R}^n)$ *with* $f=g|_\Omega$ in $D'(\Omega)$. *Here we define the quasi-norm by*

$$\|f|B^s_{pq}(\Omega)\| := \inf \|g|B^s_{pq}(\mathbb{R}^n)\| ,$$

where the infimum is taken over all g *with the above property.*

(ii) If $p\neq\infty$, $F^s_{pq}(\Omega)$ *is the collection of all* $f\in D'(\Omega)$ *such that there exists a* $g\in F^s_{pq}(\mathbb{R}^n)$ *with* $f=g|_\Omega$ in $D'(\Omega)$. *Here we define the quasi-norm by*

$$\|f|F^s_{pq}(\Omega)\| := \inf \|g|F^s_{pq}(\mathbb{R}^n)\| ,$$

where the infimum is taken over all g *with the above property.*

3.6. *Remark.* (i) $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ are quasi-Banach spaces (Banach spaces if $p,q\geq1$) and we also have the classical special cases corresponding to those mentioned in 3.4(ii).

(ii) For more information about these spaces we refer again to [Triebel 1983].

3.7. LEMMA (see [Triebel 1983, 3.2.4 and 3.3.1] for proofs). *The following* *inclusions hold in the sense of continuous embeddings:*

(i) *if* $p\neq\infty$, $B^s_{p\,\min\{p,q\}}(\Omega) \subset F^s_{pq}(\Omega) \subset B^s_{p\,\max\{p,q\}}(\Omega)$;

(ii) *if* $s_1-s_2 > \max\{0,n(1/p_1-1/p_2)\}$, $B^{s_1}_{p_1q_1}(\Omega) \subset B^{s_2}_{p_2q_2}(\Omega)$;

(iii) *if* $p_1\geq p_2$, $B^s_{p_1q}(\Omega) \subset B^s_{p_2q}(\Omega)$.

3.8. Our interest in the spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ has to do with asymptotic estimates for the Weyl numbers of embeddings I between such spaces, which will be dealt with in Chapter II. Some comments about this seem to be in order now.

3.8.1. Those estimates to be obtained later on are of the form $x_k(I) \approx f(k)$ as $k\to\infty$, where f is a function of k. In the case when the spaces involved in the embedding I coincide with classical spaces with different but equivalent quasi-norms (cf. 3.6(i) and 3.4(ii)), the same estimates also hold for the embeddings between such classical spaces with the classical quasi-norms (this follows simply by property 2.2(iii) of s-functions). In view of this, the results to be presented extend known ones involving classical spaces.

3.8.2. Another thing to note, and which will save us space and time, is that it is enough to consider $B^s_{pq}(\Omega)$ spaces in the embeddings I. This is again a consequence of property 2.2(iii) of s-functions together with 3.7(i),(ii), taking into account that, as we shall see later on, the f(k) in the estimates do not depend on the different parameters q of the spaces involved (there is an exception in Remark II.4.10, but we will see that this does not affect the form of f(k) when going to $F^s_{pq}(\Omega)$ spaces either). This is the reason why, for the rest of this section, we shall only deal with the spaces $B^s_{pq}(\Omega)$.

3.9. For each of the spaces $B^s_{pq}(\Omega)$ there is an extension operator from Ω to $\mathbb{R}^n$, that is, a bounded linear map E from $B^s_{pq}(\Omega)$ into $B^s_{pq}(\mathbb{R}^n)$ with $Ef|_{\Omega} = f$ for all $f\in B^s_{pq}(\Omega)$. If $\chi\in S(\mathbb{R}^n)$ is such that $\chi(x)=1$ if $x\in\Omega$, then χE is also an extension operator. For details about this we refer to [Triebel 1983, 3.3.4, 2.8.2 and 2.3.3].

3.10. LEMMA. $B^s_{pq}(\Omega)$ *is a* t-*Banach space for* $t=\min\{p,q,1\}$.

Proof. Since we already know that $B^s_{pq}(\Omega)$ is a quasi-Banach space, we only have to prove that the t-triangle inequality holds for the corresponding quasi-norm. Note first that for $g_1,g_2\in B^s_{pq}(\mathbb{R}^n)$ we have (with obvious modifications if $q=\infty$)

$$\|g_1+g_2|B^s_{pq}(\mathbb{R}^n)\|^t = \Big(\sum_{j=0}^{\infty} 2^{jsq}\,\|(\varphi_j g_1{}^\wedge)^\vee+(\varphi_j g_2{}^\wedge)^\vee\|_p^q\Big)^{t/q}$$

$$\le \Big(\sum_{j=0}^{\infty}(2^{jst}\,\|(\varphi_j g_1{}^\wedge)^\vee\|_p^t+2^{jst}\,\|(\varphi_j g_2{}^\wedge)^\vee\|_p^t)^{q/t}\Big)^{t/q}$$

$$\le \|g_1|B^s_{pq}(\mathbb{R}^n)\|^t+\|g_2|B^s_{pq}(\mathbb{R}^n)\|^t .$$

In view of this we obtain now for $f_1,f_2\in B^s_{pq}(\Omega)$

$$\|f_1+f_2|B^s_{pq}(\Omega)\|^t \le \inf_{\substack{g_1|_\Omega=f_1\\ g_2|_\Omega=f_2}} \|g_1+g_2|B^s_{pq}(\mathbb{R}^n)\|^t$$

$$\le \inf_{\substack{g_1|_\Omega=f_1\\ g_2|_\Omega=f_2}} (\,\|g_1|B^s_{pq}(\mathbb{R}^n)\|^t+\|g_2|B^s_{pq}(\mathbb{R}^n)\|^t\,)$$

$$= \|f_1|B^s_{pq}(\Omega)\|^t+\|f_2|B^s_{pq}(\Omega)\|^t .$$

3.11. LEMMA (see [Triebel 1983, Rem. 1.4.1/4 and 1.3.2/1]). *Given* $0<p\le q\le\infty$ *there exists a constant* c *such that* $\|f\|_q \le c\,r^{n(1/p-1/q)}\,\|f\|_p$ *for all* $r>0$ *and all* $f\in L_p^{rB_2^n}(\mathbb{R}^n)$.

3.12. LEMMA (see [Triebel 1983, p. 25]). *Let* $Q,\Gamma\subset\mathbb{R}^n$ *with* Q *compact and* Γ *such that* ${}^c\Gamma$ *is compact, and* $\tilde{p}=\min\{1,p\}$. *There exists a constant* c *such that* $\|(\varphi f^\wedge)^\vee\|_p \le c\,\|\varphi^\vee\|_{\tilde{p}}\,\|f\|_p$ *for all* $f\in L_p^Q(\mathbb{R}^n)$ *and all* $\varphi\in C_\Gamma^\infty(\mathbb{R}^n)$.

3.13. LEMMA (Representation formula — [Edmunds/Triebel 1989]). *Let* $f\in B^s_{pq}(\mathbb{R}^n)$ *and* $\psi\in S(\mathbb{R}^n)$ *be such that* $\operatorname{supp} f \subset B^n_\infty$, $\operatorname{supp}\psi \subset 2B^n_\infty$ *and* $\psi(x)=1$

for all $x\in B^n_\infty$. *Let* φ_j, $j\in\mathbb{N}_0$, *be as in 3.2 and* $\psi_\lambda(x)=\psi(2^\lambda x)$, *where* λ *depends only on* n *and is chosen so that* $(\psi-\psi_\lambda)(2^{-j-1}x).\varphi_j(x)=\varphi_j(x)$ *if* $x\in\mathbb{R}^n$ *and* $j\geq 1$. *Then for each* $N\in\mathbb{N}$

$$f = \psi f = f_N + f^N, \tag{3.4}$$

where

$$f_N = C.\psi.\Big(\sum_{m\in\mathbb{Z}^n}(\varphi_0\hat{f})^\vee(m)\,\check{\psi}(2\cdot-2m)+\sum_{j=1}^{N}\sum_{m\in\mathbb{Z}^n}(\varphi_j\hat{f})^\vee(2^{-j}m)\,(\psi-\psi_\lambda)^\vee(2^{j+1}\cdot-2m)\Big), \tag{3.5}$$

$$f^N = \psi\sum_{j=N+1}^{\infty}(\varphi_j f^\wedge)^\vee, \tag{3.6}$$

and C *is a constant that depends only on* n.

3.14. LEMMA [Edmunds/Triebel 1989]. *With the hypotheses and notation of the above lemma except that here* $f\in B^{s_1}_{p_1q_1}(\mathbb{R}^n)$, *and assuming also that* $\|f|B^{s_1}_{p_1q_1}(\mathbb{R}^n)\|\leq 1$, $s_1>s_2$, $p_1\leq p_2$ *and* $\delta:=s_1-s_2-n(1/p_1-1/p_2)>0$, *then for each* $N\in\mathbb{N}$

$$\|f^N|B^{s_2}_{p_2q_2}(\mathbb{R}^n)\| \leq c_1 2^{-N\delta}, \tag{3.7}$$

where c_1 *is a constant independent of* f *and* N.

If, in addition, for every multi-index $\gamma\in\mathbb{N}_0^n$

$$|D^\gamma\psi^\vee(x)| \leq c_{\gamma,a}2^{-\sqrt{|x|_2}}|x|_2^{-a} \quad \text{if} \quad |x|_2\geq 1, \tag{3.8}$$

where a *is a positive number at our disposal and* $c_{\gamma,a}$ *appropriate positive constants, then for each* $N\in\mathbb{N}$

$$\|f_{N,2}|B^{s_2}_{p_2q_2}(\mathbb{R}^n)\| \leq c_2 2^{-N\delta}, \tag{3.9}$$

where

$$f_{N,2}= C.\psi.\Big(\sum_{|m|_2>c(N,0)}(\varphi_0\hat{f})^\vee(m)\,\check{\psi}(2\cdot-2m)+\sum_{j=1}^{N}\sum_{|m|_2>c(N,j)}(\varphi_j\hat{f})^\vee(2^{-j}m)\,(\psi-\psi_\lambda)^\vee(2^{j+1}\cdot-2m)\Big), \tag{3.10}$$

with $c(N,j)=\max\{N^2\delta^2,2^{j+2}\sqrt{n}\}$, $j\in\mathbb{N}_0$, *and* c_2 *is a constant independent of* f *and* N.

3.15. *Remark.* The preceding result will be applied in Chapter II without any comments about the existence of a ψ such as that considered above, so let us state formally that *there exists* $\psi\in S(\mathbb{R}^n)$ *with* $\operatorname{supp}\psi \subset 2B^n_\infty$, $\psi\equiv 1$ on B^n_∞ *and verifying* (3.8) *above*: this follows for example from [Schmeisser/Triebel 1987, Def. 1.2.1/1 with $\omega(x)=\sqrt{|x|_2}+\log(|x|_2+1)$, Prop. 1.2.2/1(ii) about existence of resolutions of unity in D_ω, Rem. 1.2.2/1 on the inclusion $D_\omega\subset S_\omega$, and Def. 1.2.1/2(ii) of S_ω].

3.16. Lemma [Edmunds/Triebel 1989]. *Let* $s_1>s_2$, $p_1\le p_2$, $\delta := s_1-s_2-n(1/p_1-1/p_2)>0$ *and* I_Ω: $B^{s_1}_{p_1q_1}(\Omega)\longrightarrow B^{s_2}_{p_2q_2}(\Omega)$ *be the natural embedding (see 3.7(ii)). Then for each* $k\in\mathbb{N}$

$$a_k(I_\Omega) \le c\,k^{-\delta/n}, \tag{3.11}$$

where c *is a constant independent of* k.

3.17. Definition. *Let* φ_j, $j\in\mathbb{N}$, ψ, λ, ψ_λ *and* C *be as in 3.13. Denoting by* M_j, $j\in\mathbb{N}$, *the number of* $m\in\mathbb{Z}^n$ *such that* $|m|_2\le 2^{j+2}\sqrt{n}$, *we define for each* $j\in\mathbb{N}$

$$S_j : \{f\in B^s_{pq}(\mathbb{R}^n)\colon \operatorname{supp} f \subset B^n_\infty\} \longrightarrow l^{M_j}_p \tag{3.12a}$$

by

$$S_j f = \{(\varphi_j f^\wedge)^\vee(2^{-j}m)\}_{|m|_2\le 2^{j+2}\sqrt{n}} \tag{3.12b}$$

and

$$T_j : l^{M_j}_p \longrightarrow B^s_{pq}(\mathbb{R}^n) \tag{3.13a}$$

by

$$T_j\{a_m\}_{|m|_2\le 2^{j+2}\sqrt{n}} = C\,\psi \sum_{|m|_2\le 2^{j+2}\sqrt{n}} a_m\cdot(\psi-\psi_\lambda)^\vee(2^{j+1}\cdot -2m)\,. \tag{3.13b}$$

3.18. Lemma [Edmunds/Triebel 1989]. *For each* $j\in\mathbb{N}$, S_j *and* T_j *above are* operators *satisfying*

$$\|S_j\| \le c_1\,2^{-j(s-n/p)} \quad \text{and} \quad \|T_j\| \le c_2\,2^{j(s-n/p)}, \tag{3.14}$$

where c_1 *and* c_2 *are constants independent of* j.

3.19. LEMMA [Edmunds/Triebel a]. *Let* $s_1>s_2$ *and* $s_1-s_2>n(1/p_1-1/p_2)_+$. *For each* $j\in\mathbb{N}$ *there are operators* $A\in\mathcal{L}(l_{p_1}^{2^{jn}}, B^{s_1}_{p_1q_1}(\Omega))$ *and* $B\in\mathcal{L}(B^{s_2}_{p_2q_2}(\Omega), l_{p_2}^{2^{jn}})$ *such that*

$$I_{p_2}^{p_1}(2^{jn}) = B\, I_\Omega\, A\ , \tag{3.15}$$

where $I_\Omega : B^{s_1}_{p_1q_1}(\Omega) \longrightarrow B^{s_2}_{p_2q_2}(\Omega)$ *is the natural embedding (see 3.7(ii)),*

$$\|A\| \le c_1\, 2^{j(s_1-n/p_1)}\ , \quad \|B\| \le c_2\, 2^{-j(s_2-n/p_2)} \tag{3.16}$$

and c_1, c_2 *are constants independent of* j.

Sketch of proof. Due to translation and rescaling properties (refer to II.4.2, where the rescaling is dealt with in detail; as to the translation, it is much simpler), we can assume without loss of generality that the *unit cube* $\frac{1}{2}\overset{\circ}{B}{}^n_\infty$ is contained in Ω, and we shall indeed assume that Ω is large compared with $\frac{1}{2}\overset{\circ}{B}{}^n_\infty$ (later on, when this last assumption is necessary, we will make precise the meaning of *large* here).

Fix $j\in\mathbb{N}$. To define A we perform the following steps:

(i) consider a real valued non-vanishing $\Phi_0\in C^\infty(\mathbb{R}^n)$ with support in $\frac{1}{2}\overset{\circ}{B}{}^n_\infty$, this support being small compared with $\frac{1}{2}\overset{\circ}{B}{}^n_\infty$ (again, we will specify later on how small it should be);

(ii) set $\Phi(x)=\Phi_0(2^j x)$ and $\Phi_r(x)=\Phi(x-x^r)$, where x^r, $r=1,\dots,2^{jn}$, are the centres of the congruent cubes with side length 2^{-j} forming a tesselation of the unit cube in the usual way;

(iii) for $(\lambda_r)_r\in l_{p_1}^{2^{jn}}$ put

$$A(\lambda_r)_r = C\sum_{r=1}^{2^{jn}} \lambda_r \Phi_r\ , \tag{3.17}$$

where C is a constant to be specified afterwards (to be correct, instead of Φ_r in (3.17) we should have the restriction of Φ_r to Ω — we shall, however, make an abuse of notation and write $A(\lambda_r)_r$ as in (3.17), meaning, as it suits us better, either an element of $B^{s_1}_{p_1q_1}(\Omega)$ or of $B^{s_1}_{p_1q_1}(\mathbb{R}^n)$).

In order to estimate $\|A\|$ we use the atomic characterization of $B^{s_1}_{p_1q_1}(\mathbb{R}^n)$ given in [Frazier/Jawerth 1985, Th. 7.1] or [Triebel a, Th. 1.9.2]. Following the formulation given in the latter, since we can write

$$A(\lambda_r)_r = \sum_{k\in\mathbb{Z}^n} (\mu_k b_k(\cdot) + \sum_{\nu=0}^{\infty} \lambda_{\nu k} b_{\nu k}(\cdot)) ,$$

with $\mu_k=0$, $b_k\equiv 0$, $\forall k\in\mathbb{Z}^n$,

$$\lambda_{\nu k} = \begin{cases} C\,\delta_{\nu j}\,\lambda_r\, 2^{j(s_1-n/p_1)} & \text{if } k=[2^j x^r] \text{ for some } r\in\{1,\dots,2^{jn}\} \\ 0 & \text{otherwise} \end{cases} ,$$

$$b_{\nu k}(x) = \begin{cases} \delta_{\nu j}\, 2^{-j(s_1-n/p_1)}\, \Phi_r(x) & \text{if } k=[2^j x^r] \text{ for some } r\in\{1,\dots,2^{jn}\} \\ 0 & \text{otherwise} \end{cases} ,$$

and

$$\Big(\sum_{k\in\mathbb{Z}^n} |\mu_k|^{p_1}\Big)^{1/p_1} + \Big(\sum_{\nu=0}^{\infty}\Big(\sum_{k\in\mathbb{Z}^n} |\lambda_{\nu k}|^{p_1}\Big)^{q_1/p_1}\Big)^{1/q_1} = \Big(\sum_{r=1}^{2^{jn}} |C\lambda_r 2^{j(s_1-n/p_1)}|^{p_1}\Big)^{1/p_1} < \infty$$

(usual modification if p_1 and/or q_1 is ∞), then

$$\|A(\lambda_r)_r\,|B^{s_1}_{p_1q_1}(\Omega)\| \le \|A(\lambda_r)_r\,|B^{s_1}_{p_1q_1}(\mathbb{R}^n)\| \le c\,|C|\,2^{j(s_1-n/p_1)}|(\lambda_r)_r|_{p_1} , \tag{3.18}$$

with c independent of j, provided that b_k is a s_1-atom related to the cube $\overset{o}{B}{}^n_\infty(k,1/2)$ and $b_{\nu k}$ is a $(\overset{o}{B}{}^n_\infty(2^{-\nu}k,2^{-\nu-1}), s_1, p_1)$-atom, that is, if for fixed

$$K \ge ([s_1]+1)_+ , \quad L \ge \max\{[n(1/p_1-1)_+-s_1], -1\}$$

(L=-1 meaning that the relation involving L stated below is not required to hold) we have

$$\operatorname{supp} b_k \subset k+\tfrac{5}{2}\overset{o}{B}{}^n_\infty , \qquad |D^\alpha b_k(x)| \le 1 \qquad \text{for } |\alpha|_1 \le K ,$$

$$\operatorname{supp} b_{\nu k} \subset 2^{-\nu}k+\tfrac{5}{2}.2^{-\nu}\overset{o}{B}{}^n_\infty ,$$

$$|D^\alpha b_{\nu k}(x)| \le (2^{-\nu n})^{-1/p_1+s_1/n-|\alpha|/n} \qquad \text{for } |\alpha|_1 \le K ,$$

and

$$\int_{\mathbb{R}^n} x^\beta b_{\nu k}(x)\, dx = 0 \qquad \text{for} \quad |\beta|_1 \le L \ .$$

If the Φ_0 chosen according to (i) above does not yield all of these relations, then the function

$$\frac{\chi\,(\Phi_0 - P\Phi_0)}{\sup\limits_{\substack{x\in\mathbb{R}^n \\ |\alpha|_1 \le K}} |D^\alpha(\chi\,(\Phi_0 - P\Phi_0))|} \tag{3.19}$$

certainly does, where here $\chi \in C_0^\infty(\mathbb{R}^n)$ is a real valued function, with small support, such that $\chi \equiv 1$ on a neighbourhood of $\operatorname{supp}\Phi_0$, and $P\Phi_0$ is the orthogonal projection of Φ_0 over the subspace of the real space $L_2(\mathbb{R}^n)$ constituted by the products of χ by the polynomials of degree not exceeding L. Therefore we can redefine Φ_0 by means of (3.19), this new Φ_0 satisfying also (i) above (in particular, its support — contained, of course, in $\operatorname{supp}\chi$ — can be as small as we want, provided the initial Φ_0 has a small support too).

That is, it is possible to choose Φ_0 in accordance with (i) (with support as small as we wish) such that (3.18) holds true.

Consider then $M \in \mathbb{N}$ to be chosen later, $h \in \mathbb{R}^n$ such that $|h|_\infty = \frac{1}{4M}$, let Ω be large enough in order to contain B^n_∞, and choose Φ_0 such that $\operatorname{supp}\Phi_0 \subset \overset{\circ}{B}{}^n_\infty(0, \frac{1}{8M})$.

With this assumptions it is easily seen that

$$\operatorname{supp} \Delta^M_{-2^{-j}h}\Phi_r \subset \Omega \ , \qquad r=1, \ldots, 2^{jn} \ ,$$

which permits us to define B by

$$Bf = (\,2^{jn} f(\Delta^M_{-2^{-j}h}\Phi_r)\,)_r \quad , \qquad f \in B^{s_2}_{p_2 q_2}(\Omega) \tag{3.20}$$

(a remark of the kind made about (3.17) holds also here). Furthermore, it is then not difficult to show that

$$B\, I_\Omega\, A\, (\lambda_r)_r = C\, \|\Phi_0\|_2^2\, (-1)^M (\lambda_r)_r \ ,$$

and this proves (3.15) once we have chosen $C = \|\Phi_o\|_2^{-2}(-1)^{-M}$.

To complete the proof of the lemma, it only remains to obtain an estimate of the required form for $\|B\|$.

Set $\Psi:=\Delta_{-h}^M \Phi_o$, so that

$$\Psi^\vee(x) = (e^{ih.x}-1)^M \, \Phi_o{}^\vee(x) \ . \tag{3.21}$$

Consider fixed functions $G,H \in S(\mathbb{R}^n)$ such that $\operatorname{supp} G \subset 2B_2^n$, $\operatorname{supp} H \subset \{y \in \mathbb{R}^n : 1/4 \le |y|_2 \le 4\}$, $G \equiv 1$ on B_2^n, $H(y)=1$ if $1/2 \le |y|_2 \le 2$. Let also s_o, s_3, a and σ be fixed numbers satisfying $a > n/p_2$, $s_o + a < s_2$, $s_3 > \max\{s_2, n(1/p_2 - 1)_+\}$, $\sigma \in \mathbb{N}$ and $\sigma > a + n/2$. Then, since $\Psi^\vee \in S(\mathbb{R}^n)$,

$$\sup_{j \in \mathbb{N}} \ 2^{-js_0} \ \|\Psi^\vee(2^j \cdot) \, H \mid W_2^\sigma(\mathbb{R}^n)\| < \infty$$

and, due to (3.21), if we choose M large ($M \ge s_3 + 2\sigma$ is surely enough) we have

$$\| \, \Psi^\vee G / |\cdot|_2^{s_3} \mid W_2^\sigma(\mathbb{R}^n)\| < \infty \ .$$

Consequently there is $c>0$ such that for all $f \in B^{s_2}_{p_2 q_2}(\mathbb{R}^n)$ (with usual modification where one of the parameters is ∞)

$$\Big(\sum_{j=1}^{\infty} 2^{js_2q_2} \| \sup_{z \in \mathbb{R}^n} \frac{|(\Psi^\vee(2^{-j}\cdot) f^\wedge)^\vee(\cdot + z)|}{1 + |2^j z|_2^a} \|_{p_2}^{q_2} \Big)^{1/q_2} \le c \, \|f \mid B^{s_2}_{p_2 q_2}(\mathbb{R}^n)\| \tag{3.22}$$

(this follows from [Triebel 1988, Cor. 7, Rem. 16] or [Triebel a, Cor. 2.5.1/2], taking into account that the so-called Tauberian conditions stated there are needed only for a reverse inequality to hold).

From (3.22) it is not difficult to see that, for the same f,

$$\Big(\sum_{j=1}^{\infty} 2^{j(s_2 - n/p_2)q_2} \Big(\sum_{r=1}^{2^{jn}} |(\Psi^\vee(2^{-j}\cdot) f^\wedge)^\vee(x^r)|^{p_2} \Big)^{q_2/p_2} \Big)^{1/q_2} \le c \, \|f \mid B^{s_2}_{p_2 q_2}(\mathbb{R}^n)\| \, . \tag{3.23}$$

Note now that $(\Psi^\vee(2^{-j}\cdot) f^\wedge)^\vee(x^r) = (2\pi)^{-n/2}(f * 2^{jn}\Psi(-2^j \cdot))(x^r) = (2\pi)^{-n/2} 2^{jn} f(\Psi(2^j(\cdot - x^r))) = (2\pi)^{-n/2} 2^{jn} f(\Delta^M_{-2^{-j}h} \Phi_r))$. Hence, taking account of

(3.20) and (3.23),

$$\Big(\sum_{j=1}^{\infty} 2^{j(s_2-n/p_2)q_2}\,|Bf|_{p_2}^{q_2}\Big)^{1/q_2} \le c\,\|f|B^{s_2}_{p_2q_2}(\Omega)\| \qquad \text{for} \quad f\in B^{s_2}_{p_2q_2}(\Omega)\,,$$

with c independent of j, and the desired result follows.

4. Counting functions for eigenvalues

4.1. Let V and H be complex Hilbert spaces with $V\subset H$ in the sense of continuous embedding. Let b be a (sesquilinear) form in V (that is, defined on $V\times V$) and denote by $b+\mu$ ($\mu\in\mathbb{C}$) the form $b+\mu(\cdot,\cdot)_H$.

The form b is said to be *strongly coercive on* V if

$$\exists m>0:\ \forall u\in V,\ m\,\|u\|_V^2 \le b(u,u)\ ; \tag{4.1}$$

b is said to be *coercive on* V if, instead, the previous relation is known to hold with $b+\mu$ in place of b for some $\mu\in\mathbb{R}$.

Whenever V and H are as above and b is a continuous, Hermitian, coercive form, we say that (V,H,b) is a *variational triplet.*

4.2. **Definition.** *Given a variational triplet* (V,H,b) *and* $\lambda\in\mathbb{R}$, *we define*
$N(\lambda,V,H,b) = \inf\{\mathrm{codim}_V E : E\lhd V$ and $b-\lambda$ is strongly coercive on $E\}$.

Note that infinite values ($+\infty$) for N are allowed.

4.3. **Lemma** (see [Métivier 1977, Lem. 2.1]). *Given a variational triplet* (V,H,b), $E\lhd V$ *and* $\lambda\in\mathbb{R}$, *the following are equivalent:*

(i) $b-\lambda$ *is strongly coercive on* E;

(ii) $\exists\varepsilon>0:\ \forall u\in E,\ (b-\lambda)(u,u) \ge \varepsilon\,\|u\|_H^2$.

4.4. **Proposition** (see [Métivier 1977, Prop. 2.2]). *Given a variational triplet* (V,H,b) *and assuming that* b *is indeed strongly coercive on* V, *we have for all* $\lambda>0$

$$N(\lambda,V,H,b) = \#\{k\in\mathbb{N}_0 : d_k(S_b,H)\geq\lambda^{-1/2}\},$$

where $S_b := \{u\in V : b(u,u)\leq 1\}$ *and* $d_k(S_b,H)$ *is the* Kolmogorov kth diameter of S_b in H:

$$d_k(S_b,H) := \inf\{\sup_{u\in S_b}\inf_{v\in F}\|u-v\|_H : F\lhd H,\ \dim F\leq k\}.$$

4.5. LEMMA (see [Métivier 1977, Prop. 2.8]). *Let* (V_j,H_j,b_j), $j=1,\dots,m$, $m\in\mathbb{N}$, *be variational triplets. Consider the spaces* $V:=\prod_{j=1}^m V_j$, $H:=\prod_{j=1}^m H_j$ *endowed with the Hilbert norms given by* $\|(u_j)_j\|_V = (\sum_{j=1}^m \|u_j\|_{V_j}^2)^{1/2}$ for $(u_j)_j\in V$, *and similarly for H, and define the form* b *in* V *by* $b((u_j)_j,(v_j)_j) := \left(\prod_{j=1}^m b_j\right)((u_j)_j,(v_j)_j) := \sum_{j=1}^m b_j(u_j,v_j)$ for $(u_j)_j,(v_j)_j\in V$. *Then* (V,H,b) *is a variational triplet and*

$$N(\lambda,V,H,b) = \sum_{j=1}^m N(\lambda,V_j,H_j,b_j), \qquad \lambda\in\mathbb{R}.$$

4.6. DEFINITION. (i) *A sesquilinear Hermitian form* b *in a linear subspace* V *of a Hilbert space* H *is said to be* lower semibounded by $\gamma\in\mathbb{R}$ *if*

$$b(u,u) \geq \gamma\|u\|_H^2, \qquad \forall u\in V;$$

(ii) *the form* b, *lower semibounded by* γ, *considered above is said to be* closed *if* V *is complete when endowed with the inner product* $b-\gamma+\alpha$ *for some (and hence all)* $\alpha>0$ *(whenever we consider a closed form* b *in* V, *and* V *was not previously endowed with a topology, we will always assume that the topology of* V *is given by means of such an inner product).*

4.7. Let (V,H,b) be a variational triplet. By the coercivity of b,

$$\exists\mu\in\mathbb{R},\ m>0:\ \forall u\in V,\ m\|u\|_V^2\leq(b+\mu)(u,u); \tag{4.2}$$

on the other hand, by the continuity of b and of the embedding J of V into H, there is a M>0 such that for all $u\in V$

$$(b+\mu)(u,u) \leq b(u,u) \leq M\|u\|_V^2 \qquad \text{if } \mu<0 \tag{4.3a}$$

or $$(b+\mu)(u,u) \le M\|u\|_V^2 + \mu\|J\|\,\|u\|_V^2 \qquad \text{if} \quad \mu \ge 0 \,. \tag{4.3b}$$

Hence (4.2) and (4.3) show that $\sqrt{(b+\mu)(\cdot,\cdot)}$ *define an equivalent norm on* V, and therefore V *is complete with this norm too.* By (4.2) and the continuity of J,

$$\exists\varepsilon>0 : \forall u\in V,\ b(u,u) \ge (\varepsilon-\mu)\|u\|_H^2 \,,$$

that is, b *is lower semibounded by* $\varepsilon-\mu$. And since $b+\mu=b-(\varepsilon-\mu)+\varepsilon$ then b *is closed.*

4.8. **Proposition** (see [Edmunds/Evans 1987, pp. 175-176]). *Let* b *be a lower semibounded closed form defined in a linear subspace* V *of a Hilbert space* H *and assume that* V *is dense in* H. *Then there is a lower semibounded self-adjoint operator* A *with domain and range in* H *which has the following properties:*

(i) $\mathcal{D}(A) = \{u\in V:\ b(u,w) = (v,w)_H$ for some $v\in H$ and all $w\in V\}$; $v=Au$;

(ii) $\mathcal{D}(A)$ *is dense in* V *and in* H;

(iii) *if* $u\in V$, $v\in H$ *and* $b(u,w) = (v,w)_H$ *for all* w *in a dense subspace of* V, *then* $u\in\mathcal{D}(A)$ *and* $v=Au$.

The following result is a convenient way of bringing up together corresponding results in [Edmunds/Evans 1987, Th. II.5.2, IX.2.3 and IX.3.1]:

4.9. **Proposition.** *Let* T *be a lower semibounded self-adjoint operator in a Hilbert space* $H\neq\{0\}$ *and suppose there exists* $\lambda\in\mathbb{C}$ *such that* $(T-\lambda I)^{-1}$ *is a compact operator with domain* H. *Then* $(T-\xi I)^{-1}$ *is compact whenever it is defined and continuous with domain* H *for* $\xi\in\mathbb{C}$. *Furthermore, whenever this is not the case then* ξ *is an isolated eigenvalue of* T *(that is, it is an isolated point in the set of all eigenvalues of* T*) of finite algebraic (=geometric) multiplicity and it is possible to find an orthonormal basis for* H *consisting* *entirely of eigenvectors of* T. *Moreover, the eigenvalues of* T *can be written as a sequence* $(\lambda_j)_j$ *such that* $\lambda_1 \le \lambda_2 \le \ldots$ *(with repetition according to multiplicity), this sequence approaching* $+\infty$ *if* H *is infinite dimensional (and being finite otherwise).*

4.10. Let (V,H,b) be a variational triplet and assume also that V is dense in H and the natural embedding $J:V \longrightarrow H$ is compact. In 4.7 we deduced that b is lower semibounded and closed, so that we can use Proposition 4.8 to consider the corresponding operator A associated with b. Now, for the same $\mu \in \mathbb{R}$ considered in 4.7, $b+\mu$ is a non-negative (that is, lower semibounded by $\gamma \ge 0$) closed form in V, and the operator associated with $b+\mu$ through Proposition 4.8 is $A+\mu I$, which is, of course, non-negative. By [Edmunds/Evans 1987, Th. IV.2.8] we have then $\mathcal{D}((A+\mu I)^{1/2}) = V$. $(A+\mu I)^{1/2}$ being also self-adjoint, we know that $((A+\mu I)^{1/2}+iI)^{-1}$ and $((A+\mu I)^{1/2}-iI)^{-1}$ both exist and are continuous with domain H (see [Taylor/Lay 1980, Th. VI.8.1 and VI.8.2], and this implies, due to the compactness of J and the equality

$$((A+\mu I)^{1/2}+iI)^{-1} = J\,((A+\mu I)^{1/2}+iI)^{-1} \quad , \tag{4.4}$$

that $((A+\mu I)^{1/2}+iI)^{-1}$ is compact (of course on the right-hand side of (4.4) we considered momentarily that the target space of this operator was its range, that is, V, and used the fact that this is also a continuous operator when using the topology of V, as follows from (4.2) and Proposition 4.8(ii)). Since

$$(A+\mu I+I)^{-1} = ((A+\mu I)^{1/2}+iI)^{-1}\,((A+\mu I)^{1/2}-iI)^{-1}$$

we have also that $(A+(\mu+1)I)^{-1}$ *is a compact operator with domain* H.

Therefore, in this case and if $H \ne \{0\}$, the hypotheses of Proposition 4.9 are satisfied for T=A, and in particular we can consider the sequence $(\lambda_j)_j$ of the eigenvalues of A as defined over there.

This prepares the ground for the following

4.11. **Proposition** (see [Métivier 1977, Prop. 2.9]). *Let* (V,H,b) *be a variational triplet with* V *densely and compactly embedded in* H. *Then, for all* $\lambda\in\mathbb{R}$, $N(\lambda,V,H,b)$ *is the number of eigenvalues* λ_j *less than or equal to* λ *of the operator* A *associated with* b *(and in this case* N *will be also referred to as the* counting function for the eigenvalues of A, *or the* counting function associated with A*), that is,*

$$N(\lambda,V,H,b) = \#\{j\in\mathbb{N}: \lambda_j\le\lambda\}.$$

4.12. *Remark.* For future reference let us mention that the preceding result is proved by showing that

$$d_j(S_{b+\lambda_0},H) = (\lambda_{j+1}+\lambda_0)^{-1/2} \quad , \qquad j\in\mathbb{N}_0 ,$$

where λ_0 is a fixed number such that $b+\lambda_0$ is strongly coercive on V, and then using the characterization given in Proposition 4.4.

4.13. We are going to consider now some concrete examples of variational triplets which will be used later on in Chapter III.

Given a non-void open subset Ω of $\mathbb{R}^n$ let $H:=L_2(\Omega)$, V be a closed subspace of $H^1(\Omega):=W_2^1(\Omega)$ containing $H_0^1(\Omega)$, and b the form given by

$$(\nabla u,\nabla v)_\Omega := (\nabla u,\nabla v)_{(L_2(\Omega))^n} = \sum_{j=1}^{n}\int_\Omega D_j u\,\overline{D_j v} \quad , \qquad \forall u,v\in V, \tag{4.5}$$

the derivatives being taken in the weak sense, of course. It is a trivial exercise to see that such $(V,L_2(\Omega),(\nabla\cdot,\nabla\cdot)_\Omega)$ *is indeed a variational triplet.* We can also say that V *is dense in* $L_2(\Omega)$, for the same is true of $C_0^\infty(\Omega)$.

Given a subset S of $\mathbb{R}^n$, define

$$C_S^\infty(\mathbb{R}^n) := \{ f\in C_0^\infty(\mathbb{R}^n) : \operatorname{supp} f \subset {}^cS \} \qquad \text{and} \qquad C_S^\infty(\Omega) := \{ f|_\Omega : f\in C_S^\infty(\mathbb{R}^n) \}.$$

4.14. **Lemma.** $C_{\partial\Omega}^\infty(\Omega) = C_0^\infty(\Omega)$.

Proof. Obviously, if $g\in C_{\partial\Omega}^\infty(\Omega)$ then $g=f|_\Omega$ for some $f\in C_{\partial\Omega}^\infty(\mathbb{R}^n)$, so that

$g\in C^\infty(\Omega)$ and $\text{supp}\, g = \overline{\{x\in\Omega:g(x)\neq 0\}} = \overline{\{x\in\Omega:f(x)\neq 0\}} = \overline{\Omega\cap\{x\in\mathbb{R}^n:f(x)\neq 0\}} \subset \overline{\Omega}\cap \text{supp}\, f \subset \overline{\Omega}\cap {}^c\partial\Omega = \Omega$; $\text{supp}\, g$ ($\subset \text{supp}\, f$) is compact. Therefore $g\in C_0^\infty(\Omega)$.

Conversely, if $g\in C_0^\infty(\Omega)$ let f be the extension of g by zero outside Ω. We have $f|_\Omega=g$, $f\in C_0^\infty(\mathbb{R}^n)$ and $\text{supp}\, f = \text{supp}\, g \subset \Omega \subset {}^c\partial\Omega$, so that $g\in C_{\partial\Omega}^\infty(\Omega)$, as required.

4.15. We have then, given closed subsets S_1, S_2 of $\partial\Omega$ with $S_1\supset S_2$, that

$$C_0^\infty(\Omega) \subset C_{S_1}^\infty(\Omega) \subset C_{S_2}^\infty(\Omega) \subset C_\emptyset^\infty(\Omega) \subset H^1(\Omega) .$$

If we complete these spaces with respect to the inner product of $H^1(\Omega)$ we get, with obvious definitions,

$$H_0^1(\Omega) \subset H_{S_1}^1(\Omega) \subset H_{S_2}^1(\Omega) \subset H_\emptyset^1(\Omega) \subset H^1(\Omega)$$

(the last inclusion can indeed be an equality if $\partial\Omega$ is smooth enough, for example of class C — see [Edmunds/Evans 1987, Th. V.4.7]).

In view of 4.13 above we then have, for any closed $S\subset\partial\Omega$, that $(H_S^1(\Omega), L_2(\Omega), (\nabla\cdot,\nabla\cdot)_\Omega)$ *is a variational triplet with* $H_S^1(\Omega)$ *dense in* $L_2(\Omega)$. In particular (cf. with 4.7) this determines a densely defined lower semibounded self-adjoint operator in $L_2(\Omega)$ (which we shall denote by $-\Delta_S^\Omega$) in accordance with Proposition 4.8.

We would also like to remark that in the case $S=\partial\Omega$, $-\Delta_S^\Omega$ is the Friedrichs extension of $-\sum_{j=1}^n D_j^2$ on $C_0^\infty(\Omega)$, that is, $-\Delta_{\partial\Omega}^\Omega$ *is the Dirichlet Laplacian for* Ω.

4.16. If Ω is a non-void bounded open subset of $\mathbb{R}^n$ then $H_0^1(\Omega)$ is compactly embedded in $L_2(\Omega)$ — see [Edmunds/Evans 1987, Th. V.4.18]. Hence, by Proposition 4.11, $N(\lambda, H_0^1(\Omega), L_2(\Omega), (\nabla\cdot,\nabla\cdot)_\Omega)$ *is the number of eigenvalues (counting multiplicity) of* $-\Delta_{\partial\Omega}^\Omega$ *less than or equal to* λ.

If Ω is a non-void bounded open subset of $\mathbb{R}^n$ with boundary of class C (for example an open parallelepiped in $\mathbb{R}^n$) then $H^1(\Omega)$ is compactly embedded in $L_2(\Omega)$ – see the last reference above –, and therefore the natural embedding $H^1_S(\Omega) \longrightarrow H^1(\Omega) \longrightarrow L_2(\Omega)$ is also compact for any closed $S \subset \partial\Omega$. Hence, again by Proposition 4.11, $N(\lambda, H^1_S(\Omega), L_2(\Omega), (\nabla\cdot,\nabla\cdot)_\Omega)$ *is the number of eigenvalues (counting multiplicity) of* $-\Delta^\Omega_S$ *less than or equal to* λ.

4.17. **Proposition.** *Let* $\emptyset \neq \Omega_1 \subset \Omega_2$ *be two open subsets of* $\mathbb{R}^n$ *such that* $Vol_n(\Omega_2 \setminus \Omega_1)=0$, *and* S_1, S_2 *be two closed subsets of* $\partial\Omega_1$ *and* $\partial\Omega_2$ *respectively. Let* $\lambda \in \mathbb{R}$. *Then*

(i) *if* $S_1 \subset S_2$, $N(\lambda, H^1_{S_1}(\Omega_1), L_2(\Omega_1), (\nabla\cdot,\nabla\cdot)_{\Omega_1}) \geq N(\lambda, H^1_{S_2}(\Omega_2), L_2(\Omega_2), (\nabla\cdot,\nabla\cdot)_{\Omega_2})$;

(ii) *if* $S_1 \supset S_2$, $N(\lambda, H^1_{S_1}(\Omega_1), L_2(\Omega_1), (\nabla\cdot,\nabla\cdot)_{\Omega_1}) \leq N(\lambda, H^1_{S_2}(\Omega_2), L_2(\Omega_2), (\nabla\cdot,\nabla\cdot)_{\Omega_2})$.

Proof. (i) $1^{\underline{st}}$ step: Note that $f \in H^1_{S_2}(\Omega_2) \Rightarrow f|_{\Omega_1} \in H^1_{S_1}(\Omega_1)$.

Indeed, since it is clear (given the hypothesis) that $f|_{\Omega_1} \in H^1(\Omega_1)$, and there exists $(f_m)_m \subset C^\infty_o(\mathbb{R}^n)$ such that $\operatorname{supp} f_m \subset {}^cS_2$ and $f_m|_{\Omega_2} \xrightarrow[m\to\infty]{} f$ in $H^1(\Omega_2)$, then $\operatorname{supp} f_m \subset {}^cS_1$ (because $S_1 \subset S_2$) and $\|f_m|_{\Omega_1} - f|_{\Omega_1}\|_{H^1(\Omega_1)} \leq \|f_m|_{\Omega_2} - f\|_{H^1(\Omega_2)} \xrightarrow[m\to\infty]{} 0$, that is, $f_m|_{\Omega_1} \xrightarrow[m\to\infty]{} f|_{\Omega_1}$ in $H^1(\Omega_1)$; therefore $f|_{\Omega_1} \in H^1_{S_1}(\Omega_1)$.

$2^{\underline{nd}}$ step: Let $E_1 \lhd H^1_{S_1}(\Omega_1)$ be such that $(\nabla\cdot,\nabla\cdot)_{\Omega_1} - \lambda$ is strongly coercive on E_1. Define $E_2 := \{f \in H^1_{S_2}(\Omega_2) : f|_{\Omega_1} \in E_1\}$.

E_2 is linear subspace of $H^1_{S_2}(\Omega_2)$; furthermore, given $(f_m)_m \subset E_2$ converging to $f \in H^1_{S_2}(\Omega_2)$ we have $(f_m|_{\Omega_1})_m \subset E_1$, $f|_{\Omega_1} \in H^1_{S_1}(\Omega_1)$ and $\|f_m|_{\Omega_1} - f|_{\Omega_1}\|_{H^1(\Omega_1)} \leq \|f_m - f\|_{H^1(\Omega_2)} \xrightarrow[m\to\infty]{} 0$, that is, $f|_{\Omega_1} \in E_1$ (because E_1 is closed). Hence $E_2 \lhd H^1_{S_2}(\Omega_2)$.

It is also clear that from the strong coercivity of $(\nabla\cdot,\nabla\cdot)_{\Omega_1} - \lambda$ on E_1 we have (recall Lemma 4.3 and the hypothesis $Vol_n(\Omega_2 \setminus \Omega_1)=0$) for some $\varepsilon > 0$ and all $f \in E_2$

$$(\nabla f, \nabla f)_{\Omega_2} - (\lambda+\varepsilon)(f,f)_{L_2(\Omega_2)} = (\nabla f|_{\Omega_1}, \nabla f|_{\Omega_1})_{\Omega_1} - (\lambda+\varepsilon)(f|_{\Omega_1}, f|_{\Omega_1})_{L_2(\Omega_1)} \geq 0,$$

that is, $(\nabla\cdot,\nabla\cdot)_{\Omega_2}-\lambda$ is strongly coercive on E_2.

3rd step: Consider the map $T: H^1_{S_2}(\Omega_2)/E_2 \longrightarrow H^1_{S_1}(\Omega_1)/E_1$ given by $T(f+E_2) = f|_{\Omega_1}+E_1$. It is easy to see that T is well-defined, injective and linear, so that $\dim \mathcal{D}(T) = \dim \mathcal{R}(T)$, and thus

$$\operatorname{codim}_{H^1_{S_2}(\Omega_2)} E_2 \le \operatorname{codim}_{H^1_{S_1}(\Omega_1)} E_1 \;.$$

Therefore, by Definition 4.2, we obtain the desired conclusion.

(ii) 1st step: Note that $f\in H^1_{S_1}(\Omega_1) \Rightarrow \tilde{f}\in H^1_{S_2}(\Omega_2)$, where $\tilde{f}$ *denotes the extension by* zero *outside the domain of* f.

Indeed, since there exists (given the hypothesis) $(f_m)_m\subset C_o^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} f_m \subset {}^cS_1$ and $f_m|_{\Omega_1} \xrightarrow[m\to\infty]{} f$ in $H^1(\Omega_1)$, then $(f_m|_{\Omega_1})_m$ is a Cauchy sequence in $H^1(\Omega_1)$. Therefore (recall that $Vol_n(\Omega_2\backslash\Omega_1)=0$) $\|f_m|_{\Omega_2}-f_k|_{\Omega_2}\|_{H^1(\Omega_2)} = \|f_m|_{\Omega_1}-f_k|_{\Omega_1}\|_{H^1(\Omega_1)} \xrightarrow[m,k\to\infty]{} 0$, that is, $(f_m|_{\Omega_2})_m$ is a Cauchy sequence in $H^1(\Omega_2)$. Hence there exists $g\in H^1(\Omega_2)$ such that $f_m|_{\Omega_2} \xrightarrow[m\to\infty]{} g$ in $H^1(\Omega_2)$, and, since $\operatorname{supp} f_m \subset {}^cS_2$ (because $S_1\supset S_2$), g really belongs to $H^1_{S_2}(\Omega_2)$. Note now that, as in the first step of (i), $f_m|_{\Omega_1} \xrightarrow[m\to\infty]{} g|_{\Omega_1}$ in $H^1(\Omega_1)$, so that $f=g|_{\Omega_1}$ in $H^1(\Omega_1)$ and, due to the fact that $Vol_n(\Omega_2\backslash\Omega_1)=0$, $\tilde{f}=g$ in $H^1(\Omega_2)$, which concludes the proof that $\tilde{f}\in H^1_{S_2}(\Omega_2)$.

2nd step: Let $E_2 \lhd H^1_{S_2}(\Omega_2)$ be such that $(\nabla\cdot,\nabla\cdot)_{\Omega_2}-\lambda$ is strongly coercive on E_2. Define $E_1 := \{f\in H^1_{S_1}(\Omega_1) : \tilde{f}\in E_2\}$.

E_1 is a linear subspace of $H^1_{S_1}(\Omega_1)$ and, furthermore, given $(f_m)_m\subset E_1$ converging to $f\in H^1_{S_1}(\Omega_1)$ we have $(\widetilde{f_m})_m\subset E_2$, $\tilde{f}\in H^1_{S_2}(\Omega_2)$ and $\|\widetilde{f_m}-\tilde{f}\|_{H^1(\Omega_2)} = \|f_m-f\|_{H^1(\Omega_1)} \xrightarrow[m\to\infty]{} 0$, that is, $\tilde{f}\in E_2$ (because E_2 is closed). Hence $E_1\lhd H^1_{S_1}(\Omega_1)$.

It is also clear that from the strong coercivity of $(\nabla\cdot,\nabla\cdot)_{\Omega_2}-\lambda$ on E_2 we have (recall Lemma 4.3) for some $\varepsilon>0$ and all $f\in E_1$,

$$(\nabla f,\nabla f)_{\Omega_1}-(\lambda+\varepsilon)(f,f)_{L_2(\Omega_1)} = (\nabla\tilde{f},\nabla\tilde{f})_{\Omega_2}-(\lambda+\varepsilon)(\tilde{f},\tilde{f})_{L_2(\Omega_2)} \ge 0 \;,$$

that is, $(\nabla\cdot,\nabla\cdot)_{\Omega_1}-\lambda$ is strongly coercive on E_1.

3rd step: Consider the map $T: H^1_{S_1}(\Omega_1)/E_1 \longrightarrow H^1_{S_2}(\Omega_2)/E_2$ given by $T(f+E_1) = \tilde{f}+E_2$. It is easy to see that T is well-defined, injective and linear, so that $\dim \mathcal{D}(T) = \dim \mathcal{R}(T)$, and thus

$$\operatorname{codim}_{H^1_{S_1}(\Omega_1)} E_1 \le \operatorname{codim}_{H^1_{S_2}(\Omega_2)} E_2 \ .$$

Therefore, by Definition 4.2, we get the desired conclusion.

4.18. **Proposition.** *Let* $\emptyset \neq \Omega_1, \ldots, \Omega_m$ *be* m *pairwise disjoint open subsets of* $\mathbb{R}^n$ *and* $S_1, \ldots, S_m$ m *closed subsets of* $\partial\Omega_1, \ldots, \partial\Omega_m$ *respectively. Define* $\Omega := \bigcup_{i=1}^m \Omega_i$, $S := \bigcup_{i=1}^m S_i$, *and let* $\lambda \in \mathbb{R}$. *Then*

(i) $N(\lambda, H^1_S(\Omega), L_2(\Omega), (\nabla\cdot,\nabla\cdot)_\Omega) \le \sum_{i=1}^m N(\lambda, H^1_{S_i}(\Omega_i), L_2(\Omega_i), (\nabla\cdot,\nabla\cdot)_{\Omega_i})$.

(ii) *If, moreover,* $S_i=\partial\Omega_i$, i=1,...,m, *then a reverse inequality also holds, that is, we have equality in (i).*

Proof. First of all note that S is a closed subset of $\partial\Omega$, and that Ω is a non-void open subset of $\mathbb{R}^n$.

(i) In view of Lemma 4.5 it is clear that the first part of the proposition will be proved if we can show that

$$N(\lambda, H^1_S(\Omega), L_2(\Omega), (\nabla\cdot,\nabla\cdot)_\Omega) \le N(\lambda, \prod_{i=1}^m H^1_{S_i}(\Omega_i), \prod_{i=1}^m L_2(\Omega_i), \prod_{i=1}^m (\nabla\cdot,\nabla\cdot)_{\Omega_i}), \qquad (4.6)$$

which we shall do next.

1st step: Note that $f \in H^1_S(\Omega) \Rightarrow (f|_{\Omega_i})_i \in \prod_{i=1}^m H^1_{S_i}(\Omega_i)$ (the basic ingredient for the proof is the first step of the proof of Proposition 4.17(i)).

2nd step: Let $E \lhd \prod_{i=1}^m H^1_{S_i}(\Omega_i)$ be such that $\prod_{i=1}^m (\nabla\cdot,\nabla\cdot)_{\Omega_i}-\lambda$ is strongly coercive on E. Define $E' := \{f \in H^1_S(\Omega) : (f|_{\Omega_i})_i \in E\}$.

By an easy adaptation of the corresponding part in step 2 of the proof of Proposition 4.17(i), we have $E' \lhd H^1_S(\Omega)$.

Moreover, from the strong coercivity of $\prod_{i=1}^{m} (\nabla\cdot,\nabla\cdot)_{\Omega_i}-\lambda$ on E we can write, recalling Lemma 4.3, for some $\varepsilon>0$ and all $f \in E'$,

$$(\nabla f,\nabla f)_\Omega-(\lambda+\varepsilon)(f,f)_{L_2(\Omega)} = \sum_{i=1}^{m}(\nabla f|_{\Omega_i},\nabla f|_{\Omega_i})_{\Omega_i} - (\lambda+\varepsilon)\sum_{i=1}^{m}(f|_{\Omega_i},f|_{\Omega_i})_{L_2(\Omega_i)} \geq 0,$$

that is, $(\nabla\cdot,\nabla\cdot)_\Omega-\lambda$ is strongly coercive on E'.

3^rd^ step: Consider the map $T: H^1_S(\Omega)/E' \longrightarrow \left(\prod_{i=1}^{m} H^1_{S_i}(\Omega_i)\right)/E$ given by $T(f+E') = (f|_{\Omega_i})_i+E$. T being well-defined, injective and linear, we have $\dim \mathcal{D}(T) = \dim \mathcal{R}(T)$ and therefore

$$\operatorname{codim}_{H^1_S(\Omega)} E' \leq \operatorname{codim}_{\prod_{i=1}^{m} H^1_{S_i}(\Omega_i)} E .$$

By Definition 4.2 we then obtain (4.6) and, as a consequence, the desired conclusion.

(ii) By part (i) we already know that (4.6) holds, so in view of Lemma 4.5 we only need to prove now that

$$N(\lambda,H^1_0(\Omega),L_2(\Omega),(\nabla\cdot,\nabla\cdot)_\Omega) \geq N(\lambda,\prod_{i=1}^{m} H^1_0(\Omega_i),\prod_{i=1}^{m} L_2(\Omega_i),\prod_{i=1}^{m}(\nabla\cdot,\nabla\cdot)_{\Omega_i}). \qquad (4.7)$$

1^st^ step: Note that $(f^i)_i \in \prod_{i=1}^{m} H^1_0(\Omega_i) \Rightarrow f \in H^1_0(\Omega)$, *where* $f(x)=f^i(x)$ if $x \in \Omega_i$.

Indeed, since it is clear (given the hypothesis) that $f \in H^1(\Omega)$, and, for each $i=1,\dots,m$, there exists $(f^i_k)_k \subset C^\infty_0(\Omega_i)$ such that $f^i_k \xrightarrow[k\to\infty]{} f^i$ in $H^1(\Omega_i)$, if we define for each $k \in \mathbb{N}$, f_k in terms of f^i_k, $i=1,\dots,m$, just as we did with f above, we get $(f_k)_k \subset C^\infty_0(\Omega)$ and $\|f_k-f\|^2_{H^1(\Omega)} = \sum_{i=1}^{m}\|f^i_k-f^i\|^2_{H^1(\Omega_i)} \xrightarrow[k\to\infty]{} 0$, that is, $f_k \xrightarrow[k\to\infty]{} f$ in $H^1(\Omega)$; therefore $f \in H^1_0(\Omega)$.

2^nd^ step: Let $E \lhd H^1_0(\Omega)$ be such that $(\nabla\cdot,\nabla\cdot)_\Omega-\lambda$ is strongly coercive on E. Define $E' := \{(f^i)_i \in \prod_{i=1}^{m} H^1_0(\Omega_i) : f \in E\}$, where f is defined as above.

With the same type of techniques used in the preceding second steps, it is clear now that $E' \lhd \prod_{i=1}^{m} H_0^1(\Omega_i)$ and $\prod_{i=1}^{m} (\nabla\cdot,\nabla\cdot)_{\Omega_i} - \lambda$ is strongly coercive on E'.

3^rd^ step: Finally, the map $T : \left(\prod_{i=1}^{m} H_0^1(\Omega_i)\right)/E' \longrightarrow H_0^1(\Omega)/E$ given by $T((f^i)_i + E')$ $= f + E$ (f defined as in step 1) yields

$$\operatorname{codim}_{\prod_{i=1}^{m} H_0^1(\Omega_i)} E' \leq \operatorname{codim}_{H_0^1(\Omega)} E \ ,$$

so that, by Definition 4.2, we get (4.7) and finish the proof of the proposition.

CHAPTER II

WEYL NUMBERS OF EMBEDDINGS BETWEEN FUNCTION SPACES

1. Introduction

In recent years there has been an increased interest in the derivation of estimates for s-numbers of embedding operators between Banach spaces. This is due to the discovery, since 1978, of the so-called Weyl inequalities. In its primitive form [Weyl 1949], the Weyl inequality reads

$$\sum_{j=1}^{k} |\lambda_j(T)|^p \le \sum_{j=1}^{k} s_j(T)^p , \qquad k\in\mathbb{N} ,\ 0<p<\infty , \tag{1.1}$$

with T a compact operator on a Hilbert space H and $\lambda_j(T)$, $s_j(T)$ denoting, respectively, the eigenvalues and the s-numbers (singular numbers, at the time) of T.

In view of (1.1), information about the decay of the s-numbers can be translated to information about the decay of the eigenvalues. Besides, in a typical application T will be an integral operator taking a function space H into a subspace H_1 compactly embedded in H, so that, by I.2.2(iii),

$$s_j(T) \le s_j(H_1 \longrightarrow H) \, \|H \longrightarrow H_1\| ,$$

and information about $s_j(T)$ can be derived from the study of the s-numbers of an embedding operator between function spaces.

Starting with the work of König [1978], inequality (1.1), with a constant on the right-hand side, has been generalized for operators acting on Banach spaces, the generalizations being either by replacement of the $s_j(T)$ (which in the context of Hilbert spaces is the same for any s-function) by different s-numbers or by considering quasi-norms other than $|\cdot|_p$. In particular (1.1) holds (with a constant on the right-hand side) when H is a Banach space and

$s_j(T)=x_j(T)$, the Weyl numbers of T (to tell the whole truth, the Weyl numbers came into being partly because of the property just mentioned).

For further information on these matters the reader should consult [König 1986], [Pietsch 1987] and the references given there (the last book includes also a very interesting historical survey).

There is another reason, related to the consideration of the eigenvalues of differential operators, why the s-numbers should be of interest, but we shall delay any further comments about this until Chapter III.

In any case, since the fundamental work of Birman and Solomjak [1967], where the method of piecewise-polynomial approximation was invented to tackle the problem of estimating the approximation numbers of embeddings from a Sobolev space to a L_p-space, the estimation of s-numbers of embeddings between function spaces has gained the right to be considered as an interesting mathematical problem in its own right. Since then much work on this subject has been done, considering also Besov instead of Sobolev spaces (defined on bounded as well as unbounded domains) and other s-numbers besides the approximation numbers: Kolmogorov and, more recently, Weyl numbers.

Again, we mention only a few books where the reader can look for a more expanded list of references — [Birman/Solomjak 1980], [Triebel 1978, 4.10], [König 1986, 3.c] and [Edmunds/Evans 1987, V.6].

From what has been said it becomes clear that progress in this area is partly fed by progress in another area: the theory of function spaces (for this we refer to [Triebel 1983] and the forthcoming book [Triebel a]). Indeed, it was the fact that this theory has reached a stage of great unity and generality that led Edmunds and Triebel [1989; a] to try to get the counterpart for the asymptotic distribution of s-numbers of embeddings between function spaces of the type B^s_{pq} and F^s_{pq} (which include, in particular, the Sobolev and Besov spaces dealt with before). And there is a new ingredient here that should be mentioned: some of these spaces fall out of the realm of Banach spaces, being in general quasi-Banach spaces.

This poses immediately one question, if we are thinking about applications: is there some form of Weyl inequality in the new context?

This was not, however, the way followed by our research. Since the mentioned papers [Edmunds/Triebel 1989; a] have dealt only with the approximation numbers (as far as s-numbers are concerned, because entropy numbers are also considered there, as well as in other publications referred to above), we decided to investigate what is going on with the Weyl numbers. The reason why we considered Weyl instead of, for example, Gelfand numbers is that, if we care for a moment about a possible application to eigenvalues in the line of Weyl inequality, the Weyl numbers behave better than the Gelfand ones (which, in turn, behave better than the approximation numbers).

Part of the results to be presented in this chapter either have recently been or are about to be published [Caetano 1990; a; c].

We give next a somewhat detailed description of the sections to follow.

In 2 it may seem that we have moved to another subject, but, in fact, what is done there will be essential for the main results of the chapter. We ask therefore for some patience from the part of the reader and hope he enjoys himself with the generalization of results about sections of unit balls in sequence spaces we present there. As will be apparent, part of the inspiration for that section comes from [Meyer/Pajor 1988] and [Kanter 1977], the latter through our section I.1. We refer to the former for some references to the literature on this subject (see also [Ball 1988], [Grząślewicz 1988] and references therein).

In section 3 we derive the estimates for Weyl numbers of embeddings

$I_q^p(M): l_p^M \longrightarrow l_q^M$ that will be needed in the last section of the chapter. Once the upper estimate for the case $0<p\le q\le 2$ is clear, the results for the remaining cases are obtained from this one and other already known cases through more or less standard techniques. Although the results for $p,q\ge 1$ have already been known for some time, we treat the case pointed out above as a whole, and the decisive step is the study of the embedding $I_2^p(M)$ for $0<p\le 2$. For $p\ge 1$ this was done by Pietsch [1980a], taking advantage of a result of Bennett [1973] and (independently) Carl [1974] on (r,2)-summing operators. That method, however, doesn't seem to be extendable to $0<p<1$, and therefore we devised another approach (which also includes the case $p\ge 1$). We relate the Weyl numbers with the volume numbers, transferring thus the problem to these ones, which are estimated via the results obtained in section 2. We then get, in the complex case,

$$x_k(I_2^p(M)) \le (2/p)^{1/2}\, e^{(2p+1)/24}\, k^{1/2-1/p}.$$

For $p\in[1,2]$ the constant (that is, the value independent of k and M) on the right-hand side can be estimated from above by $\sqrt{2}\, e^{1/8}$, which is independent of p and *only slightly* worse than the $\sqrt{2}$ known to be enough in this case (see [König 1986, 2.b.10]). However, if $p\to 0^+$ it will go to infinity. We could have presented slightly more precise results in some instances of section 3, using what is known in the Banach situation — see [Linde 1986] —, but, anyway, we would not have been able to give a final answer for all the cases.

In section 4 we prove the main results of this chapter, namely Theorems 4.9, 4.12 and Corollary 4.14. These unify and extend known results of Pietsch [1980b] and Lubitz [1982]. We take advantage of the several results proved by Edmunds and Triebel [1989; a] — see our section I.3 —, where Fourier-analytical methods were used, and follow a discretization approach in order to use the results of section 3. For the upper estimates we have to derive an *ad hoc* discretization result (Proposition 4.8) — valid not only for the Weyl numbers but indeed for any s-function — which takes the place of the discretization technique of Maiorov [1975] used in [Lubitz 1982, IV.3]. The results stated in Theorem 4.9 are also going to appear in [Caetano a], where we used a non-sharp upper estimate for $x_k(I_2^p(M))$, $p\in(0,2)$, (obtained via entropy numbers) which was however enough to yield the sharp upper estimate for the corresponding Weyl numbers in function spaces.

2. Sections of unit balls

2.1. Throughout this section we deal only with real spaces and assume that the notation developed in I.2.8 (considering, obviously, the case $\mathbb{K}=\mathbb{R}$) is in force here; moreover, as a general convention, if something is typed in bold, such as **H**, and the same symbol in normal typeface (H in our example) is also used to denote a subset of $\mathbb{R}^M$, then the bold symbol denotes the inverse image by R of what is denoted by the other symbol (in our example, $\mathbf{H}=R^{-1}H$).

We begin with a result that, as far as we know, was first proved by us in [Caetano c]. Note that there is no assumption of convexity whatsoever, and that the set B in question can be really nasty.

2.2. Proposition. *Let* B *be a bounded Borel subset of* $\mathbb{R}^M$ *and* H *a* k-*dimensional subspace of* $\mathbb{R}^M$. *Assume that* $\mathrm{Vol}_{\mathbf{H}}(\mathbf{H}\cap\partial\mathbf{B})=0$. *Then*

$$\mathrm{Vol}_{\mathbf{H}}(\mathbf{H}\cap\mathbf{B}) = \lim_{\varepsilon\to 0^+}(2\varepsilon)^{k-M}\,\mathrm{Vol}_M(\mathrm{H}(\varepsilon)\cap\mathrm{B}) , \tag{2.1}$$

where $\mathrm{H}(\varepsilon) = \{ x\in\mathbb{R}^M : \ |x.R\delta_j^M| \le \varepsilon ,\ k+1\le j\le M \}$.

2.3. *Remark.* Since R is an orthogonal transformation of $\mathbb{R}^M$, hence, in particular, a homeomorphism, then $\partial\mathbf{B} = \partial R^{-1}B = R^{-1}\partial B$ and therefore

$\mathrm{Vol}_H(H\cap\partial B) = \mathrm{Vol}_k((RJ_k)^{-1}(H\cap\partial B)) = \mathrm{Vol}_k(J_k^{-1}(R^{-1}H\cap R^{-1}\partial B))$; that is, the hypothesis $\mathrm{Vol}_H(H\cap\partial B)=0$ is equivalent to $\mathrm{Vol}_k(J_k^{-1}(\mathbf{H}\cap\partial\mathbf{B}))=0$.

Proof of Proposition 2.2. The case $B=\emptyset$ or $k=M$ being trivial, we suppose $B\neq\emptyset$ and $k<M$.

Note first that, due to the fact that Lebesgue measure in $\mathbb{R}^M$ is invariant under orthogonal transformations,

$$(2\varepsilon)^{k-M}\mathrm{Vol}_M(H(\varepsilon)\cap B) = (2\varepsilon)^{k-M}\int_{\mathbb{R}^M} 1\!\!1_{\mathbf{H}(\varepsilon)\cap\mathbf{B}}(y)\,d\mathrm{Vol}_M(y), \tag{2.2}$$

Consider a sequence of non-void compact sets, $(K_m)_m$, defined in the following way:

$$K_m = \{\, x\in\mathbb{R}^M : d_\infty(x,\partial\mathbf{B}) \geq \tfrac{1}{m} \,\} \cap nB_\infty^M, \tag{2.3}$$

where $n\in\mathbb{N}$ is such that $\mathbf{B}\subset(n-1)B_\infty^M$ (this is possible because B is bounded) and d_∞ stands for the distance in $\mathbb{R}^M$ coming from the $|\cdot|_\infty$-norm.

Use of Tonelli's theorem in (2.2) gives

$$\begin{aligned}(2\varepsilon)^{k-M}\mathrm{Vol}_M(H(\varepsilon)\cap B) &= (2\varepsilon)^{k-M}\int_{\mathbb{R}^k}\int_{\mathbb{R}^{M-k}} 1\!\!1_{\mathbf{H}(\varepsilon)}(y)\,1\!\!1_{\mathbf{B}}(y)\,d\mathrm{Vol}_{M-k}(y'')\,d\mathrm{Vol}_k(y') \\ &= (2\varepsilon)^{k-M}\int_{\mathbb{R}^k} 1\!\!1_{nB_\infty^k}(y')\,E(y')\,d\mathrm{Vol}_k(y'),\end{aligned} \tag{2.4}$$

with $y=(y',y'')$, $y'=(y_1,\dots,y_k)$, $y''=(y_{k+1},\dots,y_M)$ and

$$E(y') = \int_{\mathbb{R}^{M-k}} 1\!\!1_{\mathbf{H}(\varepsilon)}(y)\,1\!\!1_{\mathbf{B}}(y)\,d\mathrm{Vol}_{M-k}(y''), \tag{2.5}$$

the last equality in (2.4) coming from the fact that $y'\notin nB_\infty^k \Rightarrow \exists\, j\in\{1,\dots,k\} : |y_j|>n \Rightarrow (y',y'')\notin nB_\infty^M \Rightarrow (y',y'')\notin\mathbf{B}$.

Since we can write $nB_\infty^M = (K_m\cap\mathbf{B})\cup(K_m\cap{}^c\mathbf{B})\cup({}^cK_m\cap nB_\infty^M)$ for every $m\in\mathbb{N}$, these being disjoint unions,

$$1\!\!1_{nB_\infty^k}(y') = 1\!\!1_{nB_\infty^M}(y',0) = 1\!\!1_{K_m\cap\mathbf{B}}(y',0) + 1\!\!1_{K_m\cap{}^c\mathbf{B}}(y',0) + 1\!\!1_{{}^cK_m\cap nB_\infty^M}(y',0).$$

Then (2.4) can be written in the form

$$(2\varepsilon)^{k-M}\,\mathrm{Vol}_M(H(\varepsilon)\cap B) = E_1 + E_2 + E_3 \tag{2.6}$$

with
$$E_1 = (2\varepsilon)^{k-M}\int_{\mathbb{R}^k} 1\!\!1_{K_m\cap\mathbf{B}}(y',0)\,E(y')\,d\mathrm{Vol}_k(y')\,,$$

$$E_2 = (2\varepsilon)^{k-M}\int_{\mathbb{R}^k} 1\!\!1_{K_m\cap{}^c\mathbf{B}}(y',0)\,E(y')\,d\mathrm{Vol}_k(y')$$

and
$$E_3 = (2\varepsilon)^{k-M}\int_{\mathbb{R}^k} 1\!\!1_{{}^cK_m\cap n B^M_\infty}(y',0)\,E(y')\,d\mathrm{Vol}_k(y')\,,$$

and from (2.6) we obtain

$$|\mathrm{Vol}_{\mathbf{H}}(H\cap B) - (2\varepsilon)^{k-M}\,\mathrm{Vol}_M(H(\varepsilon)\cap B)| \le$$
$$\le \left|\int_{\mathbb{R}^k} 1\!\!1_{\mathbf{B}}(y',0)\,d\mathrm{Vol}_k(y') - E_1\right| + E_2 + E_3\,. \tag{2.7}$$

Before going further note that, by the definition of K_m, $\{x\in\mathbb{R}^M:\ d_\infty(x,K_m)<1/m\}\cap\partial\mathbf{B} = \emptyset$.

Analysis of E_2:

In order that the integrand may be non-zero, we must have $(y',0)\in K_m\cap{}^c\mathbf{B}$. If we choose $\varepsilon<1/m$ we get, on one hand, that $(\exists\, j\in\{k+1,\dots,M\} : y_j\notin[-\varepsilon,\varepsilon]) \Rightarrow 1\!\!1_{\mathbf{H}(\varepsilon)}(y)=0$; on the other hand, $(\forall\, j\in\{k+1,\dots,M\},\, y_j\in[-\varepsilon,\varepsilon]) \Rightarrow |y-(y',0)|_\infty = |(0,y'')|_\infty \le \varepsilon < 1/m$, that is, $d_\infty(y,K_m) < 1/m$ and consequently $y\notin\partial\mathbf{B}$. In this last case we can even say that $y\notin\mathbf{B}$, for if y did belong to $\mathbf{B}$ then there would exist z in the segment joining $(y',0)$ to y that would be in $\partial\mathbf{B}$; z is of the form $(y',\lambda y'')$ with $\lambda\in(0,1)$ and thus $|z-(y',0)|_\infty < 1/m$: contradiction! This shows that $E_2=0$ if $\varepsilon<1/m$.

Analysis of E_1:

Here, in order that the integrand may be non-zero, we must have $(y',0)\in K_m\cap\mathbf{B}$. Choosing again $\varepsilon<1/m$, $(\forall\, j\in\{k+1,\dots,M\},\, y_j\in[-\varepsilon,\varepsilon]) \Rightarrow |y-(y',0)|_\infty \le \varepsilon < 1/m$, that is, $y\notin\partial\mathbf{B}$, and, reasoning in much the same way as above, we must indeed have $y\in\mathbf{B}$. Thus $1\!\!1_{\mathbf{H}(\varepsilon)}(y)\,1\!\!1_{\mathbf{B}}(y) = 1\!\!1_{\mathbf{H}(\varepsilon)}(y)$ and in the case of E_1 the function E is equal to

$$\int_{\mathbb{R}^{M-k}} \prod_{j=k+1}^{M} 1\!\!1_{[-\varepsilon,\varepsilon]}(y_j)\ d\,Vol_{M-k}(y'') = (2\varepsilon)^{M-k}. \tag{2.8}$$

Finally, in E_3 consider the upper estimate $1\!\!1_{\mathbf{B}}(y)\leq 1$ in E, so that E is bounded above by (2.8).

Putting all this together in (2.7), for all $m\in\mathbb{N}$ and $\varepsilon<1/m$ we can write

$$\begin{aligned}
|Vol_{\mathbf{H}}(H\cap B) - (2\varepsilon)^{k-M}\,Vol_{\mathbf{M}}(H(\varepsilon)\cap B)| &\leq \\
&\leq \left|\int_{\mathbb{R}^k} 1\!\!1_{\mathbf{B}}(y',0)\,dVol_k(y') - \int_{\mathbb{R}^k} 1\!\!1_{K_m\cap\mathbf{B}}(y',0)\,dVol_k(y')\right| + \\
&\qquad + \int_{\mathbb{R}^k} 1\!\!1_{{}^cK_m\cap n\mathbf{B}^M_\infty}(y',0)\,dVol_k(y') \qquad (2.9)\\
&= \int_{\mathbb{R}^k} 1\!\!1_{{}^cK_m\cap\mathbf{B}}(y',0)\,dVol_k(y') + \int_{\mathbb{R}^k} 1\!\!1_{{}^cK_m\cap n\mathbf{B}^M_\infty}(y',0)\,dVol_k(y') \\
&\leq 2\int_{\mathbb{R}^k} 1\!\!1_{{}^cK_m\cap n\mathbf{B}^M_\infty}(y',0)\,dVol_k(y')\ .
\end{aligned}$$

Observe now that $|1\!\!1_{{}^cK_m\cap n\mathbf{B}^M_\infty}(y',0)| \leq 1\!\!1_{n\mathbf{B}^M_\infty}\circ J_k(y') = 1\!\!1_{n\mathbf{B}^k_\infty}(y')$, which is integrable with respect to Vol_k. And, given $y'\in\mathbb{R}^k$ such that $(y',0)\notin\partial\mathbf{B}$, $d_\infty((y',0),\partial\mathbf{B})>0$, so that there exists $m\in\mathbb{N}$ such that $d_\infty((y',0),\partial\mathbf{B})\geq\frac{1}{m}$; thus either $(y',0)\notin nB^M_\infty$, in which case $1\!\!1_{{}^cK_m\cap n\mathbf{B}^M_\infty}(y',0)=0$, or $(y',0)\in K_m$ (recall the definition (2.3) of K_m), in which case we also have $1\!\!1_{{}^cK_m\cap n\mathbf{B}^M_\infty}(y',0)=0$, so that $\lim_{m\to\infty} 1\!\!1_{{}^cK_m\cap n\mathbf{B}^M_\infty}(y',0) = 0$. Since the set that we excluded, that is, $\{y'\in\mathbb{R}^k\colon (y',0)\in\partial\mathbf{B}\} = J_k^{-1}(\mathbf{H}\cap\partial\mathbf{B})$, has Vol_k equal to 0 (see Remark 2.3), then $1\!\!1_{{}^cK_m\cap n\mathbf{B}^M_\infty}\circ J_k \xrightarrow[m\to\infty]{} 0$ Vol_k-a.e..

We can thus use Lebesgue's dominated convergence theorem to state that

$$\int_{\mathbb{R}^k} 1\!\!1_{{}^cK_m\cap n\mathbf{B}^M_\infty}(y',0)\,dVol_k(y') \xrightarrow[m\to\infty]{} 0\ . \tag{2.10}$$

We are now in a position to conclude the proof:

Given $\eta>0$ choose m such that the integral in (2.10) is less than $\frac{\eta}{2}$, and choose $0<\varepsilon_0<1/m$. For $0<\varepsilon\leq\varepsilon_0$ we have then that (2.9) holds and therefore

$$|\mathrm{Vol}_H(H\cap B) - (2\varepsilon)^{k-M}\,\mathrm{Vol}_M(H(\varepsilon)\cap B)| < 2\cdot\frac{\eta}{2} = \eta\,.$$

2.4. **Proposition.** *If* B *is the closed unit ball corresponding to a* t*-norm* $\|\cdot\|$ *in* $\mathbb{R}^M$, $t\in(0,1]$, *then* B *satisfies the hypotheses of Proposition 2.2 for all subspaces* H *of dimension* k *of* $\mathbb{R}^M$.

Proof. Since all t-norms in $\mathbb{R}^M$ are equivalent (see [Köthe 1969, p. 151]) and it can easily be seen that B is closed in its own topology, then B is also closed in the Euclidean sense, and therefore is a Borel set. It is also clear that B is bounded, so that it only remains to prove that $\mathrm{Vol}_H(H\cap\partial B) = 0$.

Due to the continuity of the t-norm, it is easy to see that $x\in\partial B \Leftrightarrow \|x\|=1$, therefore $\partial B\subset B$ and $\lambda B\cap\partial B=\emptyset$ for all $\lambda\in[0,1)$. Then $H\cap\partial B \subset H\cap B\setminus(H\cap\lambda B)$ and

$$\mathrm{Vol}_H(H\cap\partial B) \le \mathrm{Vol}_H(H\cap B) - \mathrm{Vol}_H(H\cap\lambda B)\,, \tag{2.11}$$

since $H\cap\lambda B\subset H\cap B$, $\lambda\in[0,1)$, and $\mathrm{Vol}_H(H\cap B)<\infty$. Note now that $H\cap\lambda B=\lambda(H\cap B)$, $\lambda\neq 0$, and so $\mathrm{Vol}_H(H\cap\lambda B)=\lambda^k\mathrm{Vol}_H(H\cap B)$, $\lambda>0$. Using this in (2.11) gives

$$\mathrm{Vol}_H(H\cap\partial B) \le (1-\lambda^k)\,\mathrm{Vol}_H(H\cap B) \qquad \text{for all } \lambda\in(0,1)\,,$$

and letting $\lambda\longrightarrow 1^-$ we obtain the desired result.

2.5. **Proposition.** *Let* H *be a* k*-dimensional subspace of* $\mathbb{R}^M$, $\|\cdot\|$ *a* t*-norm in* $\mathbb{R}^M$, $t\in(0,1]$, *and* B *the corresponding closed unit ball. Let* $0<p<\infty$. *Then we have*

$$\Gamma(1+\tfrac{k}{p})\,\mathrm{Vol}_H(H\cap B) = \lim_{\varepsilon\to 0^+} (2\varepsilon)^{k-M}\int_{H(\varepsilon)} \exp(-\|x\|^p)\,d\mathrm{Vol}_M(x)\,,$$

where $H(\varepsilon)$ *has the same meaning as in (2.1).*

Proof. We follow close to the proof of Lemma II.1 of [Meyer/Pajor 1988], the main modifications being due to the fact that our B is not necessarily convex.

For $\varepsilon>0$ let

$$g(\varepsilon) := (2\varepsilon)^{k-M} \int_{H(\varepsilon)} \exp(-\|x\|^p)\ d\mathrm{Vol}_M(x)$$

$$= (2\varepsilon)^{k-M} \int_{H(\varepsilon)} \int_{\|x\|^p}^{\infty} e^{-r}\, dr\ d\mathrm{Vol}_M(x)$$

$$= (2\varepsilon)^{k-M} \int_{\mathbb{R}} e^{-r} \int_{\mathbb{R}^M} \mathbb{1}_{H(\varepsilon)}(x)\, \mathbb{1}_{[\|x\|^p,\infty)}(r)\ d\mathrm{Vol}_M(x)\ dr\ ,$$

where the last identity comes from Tonelli's theorem. Note that in the first of the iterated integrals above we can replace $\mathbb{R}$ by $\mathbb{R}^+$. For each r we now make the change of variables defined by $x=r^{1/p}y$ and get

$$g(\varepsilon) = (2\varepsilon)^{k-M}\int_{\mathbb{R}^+} e^{-r} r^{M/p} \int_{\mathbb{R}^M} \mathbb{1}_{H(\varepsilon)}(r^{1/p}y)\, \mathbb{1}_{[r\|y\|^p,\infty)}(r)\ d\mathrm{Vol}_M(y)\ dr$$

$$= (2\varepsilon)^{k-M}\int_{\mathbb{R}^+} e^{-r} r^{M/p} \int_{\mathbb{R}^M} \mathbb{1}_{H(\varepsilon r^{-1/p})}(y)\, \mathbb{1}_{B}(y)\ d\mathrm{Vol}_M(y)\ dr$$

$$= \int_{\mathbb{R}} \mathbb{1}_{(0,\infty)}(r)\, (2\varepsilon r^{-1/p})^{k-M}\, \mathrm{Vol}_M(H(\varepsilon r^{-1/p})\cap B)\, e^{-r} r^{k/p}\ dr\ . \qquad (2.12)$$

From this the result follows trivially when $k=M$, so that we suppose $k<M$ in the sequel.

The idea now is to apply Lebesgue's dominated convergence theorem, since Proposition 2.4 states that we can apply Proposition 2.2 here, and the latter asserts that for each $r\in\mathbb{R}^+$

$$\lim_{\varepsilon\to 0^+} (2\varepsilon r^{-1/p})^{k-M}\, \mathrm{Vol}_M(H(\varepsilon r^{-1/p})\cap B) = \mathrm{Vol}_H(H\cap B)\ . \qquad (2.13)$$

However, we need to prove first that the integrand in question is bounded above by an integrable function of r independent of ε.

Note that, with $y'=(y_1,\dots,y_k)$ and $y''=(y_{k+1},\dots,y_M)$, $y'\in J_k^{-1}(\mathbf{B}-(0,y'')) \Leftrightarrow (y',0)\in\mathbf{B}-(0,y'') \Leftrightarrow (y',y'')\in\mathbf{B}$. By hypothesis it follows that B is bounded, hence $\mathbf{B}$ is bounded, that is, $\mathbf{B}\subset nB_\infty^M$ for some $n\in\mathbb{N}$. Thus $(y',y'')\in\mathbf{B} \Rightarrow |y_j|\le n,\ j=1,\dots,k$, $\Rightarrow y'\in nB_\infty^k$ and $J_k^{-1}(\mathbf{B}-(0,y'')) \subset nB_\infty^k$. Therefore $\mathrm{Vol}_k(J_k^{-1}(\mathbf{B}-(0,y''))) \le n^k \mathrm{Vol}_k(B_\infty^k) = (2n)^k$.

We can then write, for every $\eta>0$ and $y=(y',y'')$,

$$\begin{aligned}
\mathrm{Vol}_M(H(\eta)\cap B) &= \mathrm{Vol}_M(\mathbf{H(\eta)}\cap\mathbf{B}) = \int_{\mathbb{R}^M} \mathbb{1}_{\mathbf{H(\eta)}}(y)\, \mathbb{1}_{\mathbf{B}}(y)\, d\,\mathrm{Vol}_M(y) \\
&= \int_{\mathbb{R}^{M-k}\times\mathbb{R}^k} \mathbb{1}_{[-\eta,\eta]^{M-k}}(y'')\, \mathbb{1}_{\mathbf{B}}(y',y'')\, d\,\mathrm{Vol}_M(y',y'') \\
&= \int_{\mathbb{R}^{M-k}} \mathbb{1}_{[-\eta,\eta]^{M-k}}(y'') \int_{\mathbb{R}^k} \mathbb{1}_{\mathbf{B}-(0,y'')}(y',0)\, d\,\mathrm{Vol}_k(y')\, d\,\mathrm{Vol}_{M-k}(y'') \\
&= \int_{\mathbb{R}^{M-k}} \mathbb{1}_{[-\eta,\eta]^{M-k}}(y'')\, \mathrm{Vol}_k(J_k^{-1}(\mathbf{B}-(0,y'')))\, d\,\mathrm{Vol}_{M-k}(y'') \\
&\le \int_{\mathbb{R}^{M-k}} \mathbb{1}_{[-\eta,\eta]^{M-k}}(y'')\, (2n)^k\, d\,\mathrm{Vol}_{M-k}(y'') = (2\eta)^{M-k}(2n)^k ,
\end{aligned}$$

and $(2\eta)^{k-M}\,\mathrm{Vol}_M(H(\eta)\cap B) \le (2n)^k$ for all $\eta>0$.

The integrand function in (2.12) is then bounded above by $\mathbb{1}_{(0,\infty)}(r)\,(2n)^k e^{-r} r^{k/p}$, which is independent of ε and integrable in $\mathbb{R}$.

Using (2.13) and Lebesgue's dominated convergence theorem we finally obtain

$$g(\varepsilon) \xrightarrow[\varepsilon\to 0^+]{} \int_{\mathbb{R}^+} \mathrm{Vol}_H(H\cap B)\, e^{-r} r^{k/p}\, dr = \mathrm{Vol}_H(H\cap B)\,\Gamma(1+\tfrac{k}{p}) .$$

As an easy adaptation of the proof of Theorem II.2 of [Meyer/Pajor 1988] we have then

2.6. **Corollary.** *Let* H *be a* k*-dimensional subspace of* $\mathbb{R}^M$ *and* $0<p<\infty$. *Then*

$$\frac{\mathrm{Vol}_H(H\cap B_p^M)}{\mathrm{Vol}_k(B_p^k)} = \lim_{\eta\to 0^+}(2\eta)^{k-M}\,\mu_p^M(H(\eta)) ,$$

where μ_p^M *is a probability measure in* $\mathbb{R}^M$ *with density* $\exp(-|2\Gamma(1+\frac{1}{p})\cdot|_p^p)$ *with respect to* Vol_M.

Proof. In Proposition 2.5 consider $t=\min\{p,1\}$ and the t-norm $|\cdot|_p$ in $\mathbb{R}^M$, and make the change of variables $x=2\Gamma(1+\frac{1}{p})y$ in the conclusion:

$$\Gamma(1+\tfrac{k}{p})\,\mathrm{Vol}_H(H\cap B_p^M) = \lim_{\varepsilon\to 0^+}(2\varepsilon)^{k-M}\int_{H((2\Gamma(1+\frac{1}{p}))^{-1}\varepsilon)} (2\Gamma(1+\tfrac{1}{p}))^M e^{-|2\Gamma(1+\frac{1}{p})y|_p^p}\, d\,\mathrm{Vol}_M(y)$$

$$\Leftrightarrow \frac{\Gamma(1+\frac{k}{p})}{(2\Gamma(1+\frac{1}{p}))^k} \mathrm{Vol}_H(H\cap B_p^M) = \lim_{\varepsilon\to 0^+} (2\varepsilon(2\Gamma(1+\tfrac{1}{p}))^{-1})^{k-M} \int_{H((2\Gamma(1+\frac{1}{p}))^{-1}\varepsilon)} e^{-|2\Gamma(1+\frac{1}{p})y|_p^p}\, d\,\mathrm{Vol}_M(y)$$

and, recalling the formula for $\mathrm{Vol}_k(B_p^k)$ in [Edmunds/Triebel 1989] (see our Table of Notation),

$$\frac{\mathrm{Vol}_H(H\cap B_p^M)}{\mathrm{Vol}_k(B_p^k)} = \lim_{\eta\to 0^+} (2\eta)^{k-M} \int_{H(\eta)} \exp(-|2\Gamma(1+\tfrac{1}{p})y|_p^p)\, d\,\mathrm{Vol}_M(y)\,.$$

That the measure μ_p^M is a probability can be proved easily by using Tonelli's theorem, reducing thus the task to the calculation of an integral in $\mathbb{R}$.

2.7. We now abandon the methods of [Meyer/Pajor 1988], since these rely on the fact that for $p\geq 1$ the function $x\longmapsto x^p$ is convex on $\mathbb{R}^+$, and we want also (and mainly) to consider the case $0<p<1$, for which this property does not hold. We are going to use, instead, the results of section I.1 about unimodality.

Observe first that the densities for μ_p^M, $0<p<\infty$, $M\in\mathbb{N}$, pointed out in Corollary 2.6 are even functions. Furthermore, if $M=1$ then it is easy to see that these probabilities satisfy Definition I.1.3 for any $p\in(0,\infty)$ and are therefore unimodal; more precisely, $\mu_p^1\in U_1$, $0<p<\infty$. On the other hand, if $p\geq 1$ we can get the unimodality without making restrictions on M, for in this case the convexity of the balls $B_p^M(0,r)$, $r\geq 0$, imply that the corresponding μ_p^M fit in Definition I.1.3; that is, $\mu_p^M\in U_M$, $1\leq p<\infty$, $M\in\mathbb{N}$.

2.8. **Proposition.** *If* $0<p<q<\infty$ *then* $\mu_q^1 \succ \mu_p^1$.

Proof. The proof for the case $1\leq p<q$ can also be seen in [Meyer/Pajor 1988].

Recall Definition I.1.4 and note that a closed, convex, symmetric subset C of $\mathbb{R}$ is a closed interval with centre at the origin. Setting apart the trivial case when C=ℝ, what we want to prove is that

$$\mu_q^1([-a,a]) \geq \mu_p^1([-a,a]) \quad \text{for all } a\geq 0. \tag{2.14}$$

We have

$$\mu_q^1([-a,a]) - \mu_p^1([-a,a]) = \int_{-a}^{a} e^{-|2\Gamma(1+\frac{1}{q})y|^q} dy - \int_{-a}^{a} e^{-|2\Gamma(1+\frac{1}{p})y|^p} dy$$
$$= 2.\Big(\int_0^a e^{-|2\Gamma(1+\frac{1}{q})y|^q} dy - \int_0^a e^{-|2\Gamma(1+\frac{1}{p})y|^p} dy\Big).$$

This is a function of the variable a, which is nil both when a=0 and $a\to\infty$, while its derivative is

$$2\,(\,e^{-|2\Gamma(1+\frac{1}{q})a|^q} - e^{-|2\Gamma(1+\frac{1}{p})a|^p}\,),$$

which is non-negative iff $(2\Gamma(1+\frac{1}{q})a)^q \leq (2\Gamma(1+\frac{1}{p})a)^p \Leftrightarrow a \leq \Big(\frac{(2\Gamma(1+\frac{1}{p}))^p}{(2\Gamma(1+\frac{1}{q}))^q}\Big)^{1/(q-p)}$, for a>0 and because p<q. Therefore

$$\mu_q^1([-a,a]) - \mu_p^1([-a,a]) \geq 0 \qquad \text{for } a\geq 0,$$

which proves (2.14) and the proposition.

2.9. **Proposition.** *If* $0<p<q<\infty$ *and* $q\geq 1$ *then* $\mu_q^M \succ \mu_p^M$ *for all* $M\in\mathbb{N}$.

Proof. We use induction on M.

The result is true when M=1, as we have just proved in the preceding proposition (even without the restriction q≥1). Suppose now it is true for $M=m\in\mathbb{N}$. Observe that $\mu_p^{m+1}=\mu_p^m\times\mu_p^1$, $\mu_q^{m+1}=\mu_q^m\times\mu_q^1$ and, as we pointed out in 2.7, $\mu_p^1\in U_1$ and $\mu_q^m\in U_m$ (because q≥1). Moreover, $\mu_q^1\succ\mu_p^1$ (it is the case M=1) and (by the induction hypothesis) $\mu_q^m\succ\mu_p^m$. Corollary I.1.9 therefore applies here to give $\mu_q^m\times\mu_q^1 \succ \mu_p^m\times\mu_p^1$.

This and the following result are weak generalizations respectively of part of Proposition I.6 and of Theorem II.2 of [Meyer/Pajor 1988], but they will be enough for our purposes, as we shall see in the following section.

2.10. COROLLARY. *Let* H *be a* k-*dimensional subspace of* $\mathbb{R}^M$ *and* $0<p<q<\infty$, $q\geq 1$. *Then*

$$\frac{\mathrm{Vol}_H(H\cap B_p^M)}{\mathrm{Vol}_k(B_p^k)} \leq \frac{\mathrm{Vol}_H(H\cap B_q^M)}{\mathrm{Vol}_k(B_q^k)} . \tag{2.15}$$

In particular,
$$\frac{\mathrm{Vol}_H(H\cap B_p^M)}{\mathrm{Vol}_k(B_p^k)} \leq 1 \qquad \textit{if} \quad 0<p\leq 2 .$$

Proof. In view of Corollary 2.6, we just need to prove that $\mu_p^M(H(\eta)) \leq \mu_q^M(H(\eta))$. Consider $H(\eta)$. This is, obviously, closed, convex and symmetric. Since, by Proposition 2.9, $\mu_q^M \succ \mu_p^M$, then, by Definition I.1.4, $\mu_q^M(H(\eta)) \geq \mu_p^M(H(\eta))$ and (2.15) follows. Consideration of the case q=2 in (2.15) gives the second part of the corollary.

3. WEYL NUMBERS IN SEQUENCE SPACES

In this section the field of scalars can either be real or complex.

3.1. PROPOSITION. *For any s-function* **s** *and all* $M\in\mathbb{N}$, *all* $k\in\{1,\dots,M\}$ *and all* $T\in\mathcal{L}(l_2,l_2^M)$,

$$s_k(T) \leq v_k(T) ,$$

where v_k *stands for the volume numbers (recall Definition I.2.10).*

Proof. Let $\varepsilon>0$ and $\rho := \frac{1}{1+\varepsilon} s_k(T)$.

We know from a result of [Pietsch 1980, p.148] that then there are $S\in\mathcal{L}(l_2^k,l_2)$, $Q\in\mathcal{L}(l_2^M,l_2^k)$ with $\|S\|,\|Q\|\leq 1$ such that $QTS = \rho I_2^2(k)$. We can therefore write

$$\frac{1}{1+\varepsilon} s_k(T) = \rho = v_k(\rho I_2^2(k)) = v_k(QTS) \leq v_k(T) , \tag{3.1}$$

where the last step is justified as follows:

The case $\rho=0$ being trivial, we assume $\rho>0$. From the fact that QTS =

$\rho I_2^2(k)$ it follows that $\dim TS l_2^k = k$. Denote then $H{:=}TS l_2^k$ and use the notation of I.2.8 to write, with the help of Lemma I.2.9,

$$v_k(QTS) = \Big(\frac{Vol_k(QTSB_2^k)}{Vol_k(B_2^k)}\Big)^{1/k'} = \Big(\frac{Vol_k(Q|_H(H\cap TSB_2^k))}{Vol_k(B_2^k)}\Big)^{1/k'}$$

$$= \Big(\frac{Vol_k(Q|_H R_k R_k^{-1}(H\cap TSB_2^k))}{Vol_k(B_2^k)}\Big)^{1/k'}$$

$$= |\det(Q|_H R_k)|^{1/k}\Big(\frac{Vol_H(H\cap TSB_2^k)}{Vol_k(B_2^k)}\Big)^{1/k'}$$

$$\le \|Q|_H R_k\|\Big(\frac{Vol_H(H\cap \|S\|TB_2)}{Vol_k(B_2^k)}\Big)^{1/k'} \le v_k(T),$$

taking into account that $\|R_k\|,\|Q\|,\|S\| \le 1$.

Letting $\varepsilon\to 0^+$ in (3.1), we obtain the desired result.

We recall that, with $p,q\in(0,\infty]$ and $M\in\mathbb{N}$, $I_q^p(M)$ denotes the natural embedding $l_p^M \longrightarrow l_q^M$.

3.2. **Corollary.** *For all* $M\in\mathbb{N}$, *all* $k\in\{1,\dots,M\}$ *and all* $p\in(0,\infty]$ *we have*

$$x_k(I_2^p(M)) \le v_k(I_2^p(M)) ,$$

where x_k *stands for the Weyl numbers (see I.2.5.3).*

Proof. $$x_k(I_2^p(M)) = \sup\{\, a_k(I_2^p(M)A) : A\in\mathcal{L}(l_2,l_p^M),\ \|A\|\le 1\,\}$$

$$\le \sup\{\, v_k(I_2^p(M)A) : A\in\mathcal{L}(l_2,l_p^M),\ \|A\|\le 1\,\} \le v_k(I_2^p(M)) ,$$

since $Vol_H(H\cap I_2^p(M)AB_2) = Vol_H(H\cap AB_2) \le Vol_H(H\cap B_p^M)$, the inequality coming from the fact that $x\in AB_2 \Rightarrow x=Ay$ for some $y\in B_2 \Rightarrow |x|_p \le \|A\|\,|y|_2 \le 1$.

3.3. **Proposition.** *For all* $M\in\mathbb{N}$, *all* $k\in\{1,\dots,M\}$ *and all* $p\in(0,2]$,

$$x_k(I_2^p(M)) \le (\tfrac{2}{p})^{1/2}\, e^{(p+2)/12}\, k^{1/2-1/p} .$$

Proof. We deal only with the complex case (the real case is even simpler).

From Corollary 3.2 and Proposition I.2.12 (and using the convention made in this one) we have

$$x_k(I_2^p(M)) \le 2^{1/p-1/2} \sup\Big\{ \Big(\frac{\mathrm{Vol}_H(H\cap B_p^{2M}(\mathbb{R}))}{\mathrm{Vol}_{2k}(B_2^{2k}(\mathbb{R}))}\Big)^{1/(2k)} : H \lhd \ell_2^{2M}(\mathbb{R}) \ , \ \dim H = 2k \Big\}.$$

Using Corollary 2.10 we can thus write

$$x_k(I_2^p(M) \le 2^{1/p-1/2}\Big(\frac{\mathrm{Vol}_{2k}(B_p^{2k}(\mathbb{R}))}{\mathrm{Vol}_{2k}(B_2^{2k}(\mathbb{R}))}\Big)^{1/(2k)}$$

and, using the expression (3.1/2) of [Edmunds/Triebel 1989] for these Vol_{2k} (see our Table of Notation),

$$x_k(I_2^p(M)) \le 2^{1/p-1/2}\Big((\tfrac{2}{p})^{(2k-1)/2}\,(2k)^{(1/2-1/p)2k}\,e^{p.2k/12+1/(12k)}\Big)^{1/(2k)}$$

$$\le (\tfrac{2}{p})^{1/2}\,e^{(2p+1)/24}\,k^{1/2-1/p}\,.$$

3.4. **Corollary.** *For all* $M\in\mathbb{N}$, *all* $k\in\{1,\dots,M\}$ *and all* $0<p<q\le 2$,

$$x_k(I_q^p(M)) \le (\tfrac{2}{p})^{\vartheta/2}\ e^{(p+2)\vartheta/12}\ k^{1/q-1/p}\,,$$

where $\vartheta = (\frac{1}{p}-\frac{1}{q})/(\frac{1}{p}-\frac{1}{2})$.

Proof. As in [König 1986, p. 102] — see also [Caetano a] — we use an interpolation argument, as follows. Since $\frac{2}{\vartheta q}>1$ and $\frac{(1-\vartheta)q}{p}+\frac{\vartheta q}{2}=1$, Hölder's inequality gives $|x|_q \le |x|_2^{\vartheta}\,|x|_p^{1-\vartheta}$ for $x\in\mathbb{K}^M$, from which follows (use, for example, the characterization given in Proposition I.2.6(c))

$$x_k(I_q^p(M)) \le (x_k(I_2^p(M)))^{\vartheta}\,.$$

Application of the preceding proposition then gives the required result.

3.5. **Proposition.** *Assuming that* $k\le M/2$,

$$x_k(l_q^p(M)) \approx \begin{cases} 1 & \text{if } 2\le p\le q\le\infty \\ k^{1/q-1/p} & \text{if } 0<p\le q\le 2 \\ k^{1/2-1/p} & \text{if } 0<p\le 2\le q\le\infty \\ M^{1/q-1/p} & \text{if } 0<q\le p\le 2 \end{cases} \qquad \text{as } k,M\longrightarrow\infty .$$

Proof. (i) Case $2\le p\le q\le\infty$.

The upper estimate $x_k(l_q^p(M))\le 1$ is trivial. For the lower bound note that (using [Pietsch 1980, 11.7.4 and 11.11.8])

$$x_k(l_q^p(M)) \ge a_k(l_q^2(M)) = a_k(l_2^{q'}(M)) \ge a_k(l_2^1(M)) \ge \left(\frac{M-k+1}{M}\right)^{1/2},$$

where q' is the conjugate exponent of q. Since $k\le M/2$ then $x_k(l_q^p(M))\ge 1/\sqrt{2}$.

(ii) Case $0<p\le q\le 2$.

The upper estimate follows from Corollary 3.4 and easy operator quasi-norm estimates when p=q. Once this is established we can write, with the help of Proposition I.2.4 and the multiplicativity of the Weyl numbers (see Proposition I.2.7),

$$(2k)^{1/2-1/p} \le x_{2k}(l_2^p(M)) \le x_k(l_2^q(M))\, x_k(l_q^p(M)) \le c_1 k^{1/2-1/q}\, x_k(l_q^p(M))$$

and $x_k(l_q^p(M)) \ge c_2 k^{1/q-1/p}$, as required.

(iii) Case $0<p\le 2\le q\le\infty$.

The upper estimate follows easily from Proposition 3.3: $x_k(l_q^p(M)) \le \|l_q^2(M)\|\, x_k(l_2^p(M)) \le c\,k^{1/2-1/p}$. On the other hand, using (i),

$$c_1 \le x_k(l_q^2(2k)) \le x_k(l_q^p(2k))\, \|l_p^2(2k)\| \le x_k(l_q^p(M))\,(2k)^{1/p-1/2}$$

and $x_k(l_q^p(M)) \ge c_2 k^{1/2-1/p}$, as claimed.

(iii) Case $0<q\le p\le 2$.

Again the upper estimate is easy — this time just use an operator quasi-norm estimate. As to the lower one, using Propositions I.2.4, 3.3 and the multiplicativity of the Weyl numbers, we have

$$M^{1/2-1/p} \le x_M(I_2^p(M)) \le x_{M-k+1}(I_2^q(M))\, x_k(I_q^p(M)) \le c_1\,(M-k+1)^{1/2-1/q}\; x_k(I_q^p(M))$$

and, since $k \le M/2$, $x_k(I_q^p(M)) \ge c_2\, M^{1/q-1/p}$.

3.6. *Remark.* It follows from the preceding proof that the upper estimates in the above proposition hold true for any $k \in \mathbb{N}$ (cf. also with Definition I.2.2(iv)).

3.7. **Proposition** (see [König 1986, 3.c.4]). *Let* $2 \le q < p \le \infty$. *There exists* c *such that for all* $k, M \in \mathbb{N}$

$$x_k(I_q^p(M)) \le c\,(\tfrac{M}{k})^{(1/q-1/p)/(1-2/p)} .$$

3.8. **Proposition.** *Let* $0 < q \le 2 < p \le \infty$. *There exists* c *such that for all* $k, M \in \mathbb{N}$

(i) $x_k(I_q^p(M)) \ge c\, M^{1/q-1/2}$ *if* $k \le M/2$;

(ii) $x_k(I_q^p(M)) \ge c\, M^{1/q-1/p}$ *if* $k \le M^{2/p}$.

Proof. (i) Using the multiplicativity of the Weyl numbers together with Definition I.2.2(v) and Proposition 3.3, we can write

$$1 = x_M(I_2^2(M)) \le x_{M-k+1}(I_2^q(M))\, x_k(I_q^p(M))\, \|I_p^2(M)\| \le c_1\,(M-k+1)^{1/2-1/q}\; x_k(I_q^p(M))$$

and, using the hypothesis $k \le M/2$, $x_k(I_q^p(M)) \ge c_2\, M^{1/q-1/2}$.

(ii) We already know from [Heinrich/Linde 1984] that

$$x_k(I_2^p(M)) \ge c\, M^{1/2-1/p} \quad \text{if } k \le M^{2/p}, \tag{3.2}$$

which enables us the use of an interpolation argument, as follows (assuming that $0 < q < 2 < p \le \infty$, with obvious modifications if $p = \infty$). Define $\vartheta := \frac{1/2-1/p}{1/q-1/p}$, consider $\frac{q}{2\vartheta}$, which is greater than 1, and check that $\frac{2\vartheta}{q} + \frac{2(1-\vartheta)}{p} = 1$, which justifies the following application of Hölder's inequality:

$$\Big(\sum_{j=1}^{M}|x_j|^2\Big)^{1/2} = \Big(\sum_{j=1}^{M}|x_j|^{2\vartheta+2(1-\vartheta)}\Big)^{1/2} \le \Big(\sum_{j=1}^{M}|x_j|^q\Big)^{\vartheta/q}\Big(\sum_{j=1}^{M}|x_j|^p\Big)^{(1-\vartheta)/p},$$

that is, $|x|_2 \le |x|_q^{\vartheta}|x|_p^{1-\vartheta}$ for $x\in\mathbb{K}^M$. Hence $x_k(I_2^p(M)) \le (x_k(I_q^p(M)))^{\vartheta}$, and use of (3.2) in this inequality completes the proof.

4. WEYL NUMBERS IN FUNCTION SPACES

4.1. Throughout this section we use the conventions stated at the very beginning of I.3, which we briefly recall here: Ω stands for a non-empty, open, bounded C^∞-domain in the Euclidean space $\mathbb{R}^n$; $s,s_1,s_2\in\mathbb{R}$, $p,p_1,p_2,q,q_1,q_2\in(0,\infty]$; all function spaces are complex.

Besides, we set

$$\delta := s_1-s_2-n(1/p_1-1/p_2)_+ > 0\ ,$$

so that there exists the embedding

$$I(\Omega) : B^{s_1}_{p_1q_1}(\Omega) \longrightarrow B^{s_2}_{p_2q_2}(\Omega)$$

(cf. with Lemma I.3.7(ii)). $I(\Omega)$ will sometimes be written in the form $I^{s_1p_1q_1}_{s_2p_2q_2}(\Omega)$ — or simply $I^{111}_{222}(\Omega)$, which will be different from $I^{111}_{222}(\Omega)$ — when there is danger of confusion, and a similar procedure will be carried out with all mappings under consideration (for R, E and $J_r(\Omega)$ defined below, we simplify a little more the notation: since in this case the same parameters s,p,q appear both in the domain and in the target space, we shall omit the superscripts).

4.2. Our goal is to obtain two-sided estimates for $x_k(I(\Omega))$ as $k\to\infty$, which will be achieved by deriving separately upper and lower bounds for these Weyl numbers.

We begin with the upper estimates and *we may assume without loss of* *generality that* $\overline{\Omega} \subset \overset{\circ}{B}{}^n_\infty$. This assertion was mentioned in passing in [Edmunds/Triebel 1989], but we would like to be more detailed about it: we claim that given $r \in \mathbb{Z}$ and a C^∞-domain Ω as described above, it is true that $2^r\Omega$ is also such a domain and that there exist operators $J_r(\Omega)$ from $B^s_{pq}(\Omega)$ to $B^s_{pq}(2^r\Omega)$ (for all s,p,q as detailed in 4.1) such that $I^{111}_{222}(\Omega) = J_{222,-r}(2^r\Omega)\, I^{111}_{222}(2^r\Omega)\, J_{111,r}(\Omega)$; from this, any upper estimate for $x_k(I(2^r\Omega))$ will hold true for $x_k(I(\Omega))$ too (recall Definition I.2.2(iii)); and since for each Ω it is always possible to find $r \in \mathbb{Z}$ such that $\overline{2^r\Omega} \subset \overset{\circ}{B}{}^n_\infty$, we can restrict our attention to domains Ω with closure in $\overset{\circ}{B}{}^n_\infty$.

Let us prove our claim then.

First of all, it is easy to see that $2^r\Omega$ is a non-void, open, bounded C^∞-domain, for the same is true of Ω.

Now *define* $J_r(\Omega): \; B^s_{pq}(\Omega) \longrightarrow B^s_{pq}(2^r\Omega)$ by $(J_r(\Omega)f)(\Psi) = f(\Psi(2^r\cdot))$, $\forall \Psi \in C^\infty_0(2^r\Omega)$. We must, of course, prove that this is indeed *well-defined*. Assuming this for a moment, the linearity of $J_r(\Omega)$ follows immediately, and, as a consequence, an estimate of the sort $\|J_r(\Omega)f\,|\,B^s_{pq}(2^r\Omega)\| \le c\,\|f\,|\,B^s_{pq}(\Omega)\|$, with c independent of f, will then also prove the continuity of $J_r(\Omega)$ — cf. I.2.1.

Recall that

$$\|f\,|\,B^s_{pq}(\Omega)\| = \inf\{\, \|F\,|\,B^s_{pq}(\mathbb{R}^n)\| : F \in B^s_{pq}(\mathbb{R}^n),\ F|_\Omega = f \text{ in } D'(\Omega)\,\}.$$

For each $F \in B^s_{pq}(\mathbb{R}^n)$ such that $F|_\Omega = f$ define G by $G(\Psi) = F(\Psi(2^r\cdot))$, $\forall \Psi \in S(\mathbb{R}^n)$ — note that $\Psi(2^r\cdot) \in S(\mathbb{R}^n)$ too. G is obviously linear and, since it is the composition of continuous maps, it is also continuous, that is, $G \in S'(\mathbb{R}^n)$. It is also easily seen that the map $F \longmapsto G$ is injective. Furthermore, given $\Psi \in C^\infty_0(2^r\Omega)$ and denoting by $\tilde{\Psi}$ its extension by 0 outside $2^r\Omega$, we have $G|_{2^r\Omega}(\Psi) = G(\tilde{\Psi}) = F(\tilde{\Psi}(2^r\cdot)) = F(\widetilde{\Psi(2^r\cdot)}) = F|_\Omega(\Psi(2^r\cdot)) = f(\Psi(2^r\cdot)) = (J_r(\Omega)f)(\Psi)$, from which follows in particular that if we prove that $G \in B^s_{pq}(\mathbb{R}^n)$ then $J_r(\Omega)$

is well-defined and the mapping $\{H\in B^s_{pq}(\mathbb{R}^n) : H|_\Omega = f\} \longrightarrow \{H\in B^s_{pq}(\mathbb{R}^n) : H|_{2^r\Omega} = J_r(\Omega)f\}$ given by $F\longmapsto G$ defined above is bijective. Moreover, if we prove that $G\in B^s_{pq}(\mathbb{R}^n)$ by means of an estimate of the form $\|G|B^s_{pq}(\mathbb{R}^n)\| \le c\,\|F|B^s_{pq}(\mathbb{R}^n)\|$, with c independent of F and f, then we shall get the required continuity of $J_r(\Omega)$.

Let us therefore prove that an inequality of the last type holds true.

Note first that (recalling the definition of φ_j in I.3.2), for $\Psi\in S(\mathbb{R}^n)$, $G^\wedge(\Psi) = 2^{-rn}F^\wedge(\Psi(2^{-r}\cdot))$, $(\varphi_j G^\wedge)(\Psi) = 2^{-rn}(\varphi_j(2^{-r}\cdot)F^\wedge)(\Psi(2^{-r}\cdot))$ and $(\varphi_j G^\wedge)^\vee(\Psi) = (\varphi_j(2^{-r}\cdot)F^\wedge)^\vee(\Psi(2^r\cdot))$. Hence

$$(\varphi_j G^\wedge)^\vee(\Psi) = \int_{\mathbb{R}^n} (\varphi_j(2^{-r}\cdot)F^\wedge)^\vee(x)\,\Psi(2^r x)\,dx = \int_{\mathbb{R}^n} 2^{-rn}(\varphi_j(2^{-r}\cdot)F^\wedge)^\vee(2^{-r}y)\,\Psi(y)\,dy\ ,$$

that is,

$$(\varphi_j G^\wedge)^\vee(x) = 2^{-rn}(\varphi_j(2^{-r}\cdot)F^\wedge)^\vee(2^{-r}x)\ , \quad x\in\mathbb{R}^n, \tag{4.1}$$

and

$$\|(\varphi_j G^\wedge)^\vee\|_p = 2^{rn(1/p-1)}\,\|(\varphi_j(2^{-r}\cdot)F^\wedge)^\vee\|_p \tag{4.2}$$

(with $1/p=0$ if $p=\infty$, as usual).

We distinguish now several cases.

(i) $j\in\mathbb{N} \wedge j+r>0$.

Here we have $\varphi_j(2^{-r}\cdot) = \varphi_{j+r}$ and, from (4.2),

$$\|(\varphi_j G^\wedge)^\vee\|_p = 2^{rn(1/p-1)}\,\|(\varphi_{j+r}F^\wedge)^\vee\|_p\ . \tag{4.3}$$

(ii) $j\in\mathbb{N} \wedge j+r<0$.

In this case $\varphi_j(2^{-r}\cdot) = \varphi(2^{-j-r}\cdot) - \varphi(2^{-j-r+1}\cdot) = (\varphi(2^{-j-r}\cdot) - \varphi(2^{-j-r+1}\cdot))\,\varphi$, so that we can write $(\varphi_j(2^{-r}\cdot)F^\wedge)^\vee = (\varphi_j(2^{-r}\cdot)\varphi F^\wedge)^\vee = (\varphi_j(2^{-r}\cdot)((\varphi F^\wedge)^\vee)^\wedge)^\vee$. Since $F\in B^s_{pq}(\mathbb{R}^n)$ then $(\varphi F^\wedge)^\vee \in L_p^{2B_2^n}(\mathbb{R}^n)$ and Lemma I.3.12 applies to yield

$$\|(\varphi_j(2^{-r}\cdot)((\varphi F^\wedge)^\vee)^\wedge)^\vee\|_p \le c\,\|\varphi_j(2^{-r}\cdot)^\vee\|_{\tilde{p}}\,\|(\varphi F^\wedge)^\vee\|_p\ ,$$

where c depends only on n and p. From this and (4.2) we get

$$\begin{aligned} \|(\varphi_j G^\wedge)^\vee\|_p &= 2^{rn(1/p-1)} \|(\varphi_j(2^{-r}\cdot)F^\wedge)^\vee\|_p \\ &\le c\, 2^{rn(1/p-1)} \|\varphi_j(2^{-r}\cdot)^\vee\|_{\tilde{p}} \|(\varphi F^\wedge)^\vee\|_p \qquad (4.4) \\ &= c_j \|(\varphi F^\wedge)^\vee\|_p \quad \text{say}. \end{aligned}$$

(iii) $j\in\mathbb{N} \wedge j+r=0$.

Here $\varphi_j(2^{-r}\cdot) = \varphi - \varphi(2\cdot)$, therefore $(\varphi_j(2^{-r}\cdot)F^\wedge)^\vee = ((\varphi - \varphi(2\cdot))F^\wedge)^\vee = (\varphi F^\wedge)^\vee - (\varphi(2\cdot)F^\wedge)^\vee$ and $\|(\varphi_j(2^{-r}\cdot)F^\wedge)^\vee\|_p \le C\,(\|(\varphi F^\wedge)^\vee\|_p + \|(\varphi(2\cdot)\varphi F^\wedge)^\vee\|_p) \le C\,(\|(\varphi F^\wedge)^\vee\|_p + c\,\|\varphi(2\cdot)^\vee\|_{\tilde{p}} \|(\varphi F^\wedge)^\vee\|_p) = C\,(1 + c\,\|\varphi(2\cdot)^\vee\|_{\tilde{p}})\, \|(\varphi F^\wedge)^\vee\|_p$, where in the last inequality we have used Lemma I.3.12 again. From this and (4.2) we get

$$\begin{aligned} \|(\varphi_j G^\wedge)^\vee\|_p &= 2^{rn(1/p-1)} \|(\varphi_j(2^{-r}\cdot)F^\wedge)^\vee\|_p \\ &\le C\, 2^{rn(1/p-1)} (1 + c\,\|\varphi(2\cdot)^\vee\|_{\tilde{p}})\, \|(\varphi F^\wedge)^\vee\|_p \qquad (4.5) \\ &= c_0 \|(\varphi F^\wedge)^\vee\|_p \quad \text{say}. \end{aligned}$$

(iv) $j=0 \wedge r<0$.

We have $\varphi_j(2^{-r}\cdot) = \varphi(2^{-r}\cdot) = \varphi(2^{-r}\cdot)\,\varphi$, $\|(\varphi_j(2^{-r}\cdot)F^\wedge)^\vee\|_p \le c\,\|\varphi(2^{-r}\cdot)^\vee\|_{\tilde{p}} \|(\varphi F^\wedge)^\vee\|_p$ (by using again Lemma I.3.12) and, with the help of (4.2),

$$\begin{aligned} \|(\varphi_j G^\wedge)^\vee\|_p &\le c\, 2^{rn(1/p-1)} \|\varphi(2^{-r}\cdot)^\vee\|_{\tilde{p}}\, \|(\varphi F^\wedge)^\vee\|_p \\ &= c_{-1} \|(\varphi F^\wedge)^\vee\|_p \quad \text{say}. \qquad (4.6) \end{aligned}$$

(v) $j=0 \wedge r\ge 0$.

In this case we can write $\varphi_j(2^{-r}\cdot) = \sum_{k=0}^{r} \varphi_k$, thus $(\varphi_j(2^{-r}\cdot)F^\wedge)^\vee = \sum_{k=0}^{r} (\varphi_k F^\wedge)^\vee$ and, by (4.2) and with $\tilde{p} = \min\{1,p\}$, $\|(\varphi_j G^\wedge)^\vee\|_p^{\tilde{p}} = 2^{rn(1/p-1)\tilde{p}}\, \|(\varphi_j(2^{-r}\cdot)F^\wedge)^\vee\|_p^{\tilde{p}} \le 2^{rn(1/p-1)\tilde{p}} \sum_{k=0}^{r} \|(\varphi_k F^\wedge)^\vee\|_p^{\tilde{p}}$, that is,

$$\|(\varphi_j G^\wedge)^\vee\|_p \le (r+1)^{1/p}\, 2^{rn(1/p-1)} \sum_{k=0}^{r} \|(\varphi_k F^\wedge)^\vee\|_p = c_{-2} \sum_{k=0}^{r} \|(\varphi_k F^\wedge)^\vee\|_p \quad \text{say}. \qquad (4.7)$$

Now we put things together.

If $r \geq 0$ we obtain, using (4.3) and (4.7),

$$\|G|B^s_{pq}(\mathbb{R}^n)\| \leq \Big((c_{-2} \sum_{k=0}^{r} \|(\varphi_k F^\wedge)^\vee\|_p)^q + \sum_{j=1}^{\infty} 2^{jsq}\, 2^{rn(1/p-1)q} \|(\varphi_{j+r}F^\wedge)^\vee\|_p^q \Big)^{1/q}$$

$$\leq \Big(c_{-2}^q (r+1)^q \max\{1,2^{-rsq}\} \sum_{k=0}^{r} \min\{1,2^{rsq}\} \|(\varphi_k F^\wedge)^\vee\|_p^q +$$

$$+ 2^{-rsq}\, 2^{rn(1/p-1)q} \sum_{j=1}^{\infty} 2^{(j+r)sq} \|(\varphi_{j+r}F^\wedge)^\vee\|_p^q \Big)^{1/q}$$

$$\leq \max\{ c_{-2} (r+1) \max\{1,2^{-rs}\},\ 2^{-rs}\, 2^{rn(1/p-1)} \}\ \|F|B^s_{pq}(\mathbb{R}^n)\|$$

(obvious modifications if $q=\infty$).

If $r<0$ we use (4.3) to (4.6) to write (setting $\sum_{j=1}^{-r-1} \ldots := 0$ if $r=-1$)

$$\|G|B^s_{pq}(\mathbb{R}^n)\| \leq \Big(c_{-1}^q \|(\varphi F^\wedge)^\vee\|_p^q + \sum_{j=1}^{-r-1} 2^{jsq}\, c_j^q \|(\varphi F^\wedge)^\vee\|_p^q + 2^{-rsq} c_0^q \|(\varphi F^\wedge)^\vee\|_p^q +$$

$$+ \sum_{j=-r+1}^{\infty} 2^{jsq}\, 2^{rn(1/p-1)q} \|(\varphi_{j+r}F^\wedge)^\vee\|_p^q \Big)^{1/q}$$

$$\leq \Big((c_{-1}^q + \sum_{j=1}^{-r-1} 2^{jsq} c_j^q + 2^{-rsq} c_0^q) \|(\varphi F^\wedge)^\vee\|_p^q +$$

$$+ 2^{-rsq}\, 2^{rn(1/p-1)q} \sum_{j=-r+1}^{\infty} 2^{(j+r)sq} \|(\varphi_{j+r}F^\wedge)^\vee\|_p^q \Big)^{1/q}$$

$$\leq \max\{ (c_{-1}^q + \sum_{j=1}^{-r-1} 2^{jsq} c_j^q + 2^{-rsq} c_0^q)^{1/q},\ 2^{-rs+rn(1/p-1)} \}\ \|F|B^s_{pq}(\mathbb{R}^n)\|$$

(obvious modifications if $q=\infty$).

To finish off the proof of our claim, just note that the identity $I^{111}_{222}(\Omega) = J_{222,-r}(2^r\Omega)\, I^{111}_{222}(2^r\Omega)\, J_{111,r}(\Omega)$ is trivially true.

4.3. Suppose for the moment that $p_1 \leq p_2$ and note that, under this assumption, $B^{s_1}_{p_1q_1}(\mathbb{R}^n)$ *is continuously embedded in* $B^{s_2}_{p_2q_2}(\mathbb{R}^n)$. In fact, given $f \in B^{s_1}_{p_1q_1}(\mathbb{R}^n)$ we have $(\varphi_j f^\wedge)^\vee \in L_{p_1}^{2^{1+j}B_2^n}(\mathbb{R}^n)$, $j \in \mathbb{N}_0$ (with φ_j as in I.3.2), so that Lemma I.3.11 applies to yield $\|(\varphi_j f^\wedge)^\vee\|_{p_2} \leq c\, 2^{(1+j)n(1/p_1-1/p_2)} \|(\varphi_j f^\wedge)^\vee\|_{p_1}$ with c depending only on n,p_1,p_2; therefore

$$\|f|B^{s_2}_{p_2q_2}(\mathbb{R}^n)\| \leq \Big(\sum_{j=0}^{\infty} 2^{js_2q_2} c^{q_2}\, 2^{(1+j)n(1/p_1-1/p_2)q_2} \|(\varphi_j f^\wedge)^\vee\|_{p_1}^{q_2} \Big)^{1/q_2}$$

$$\leq \begin{cases} \left(\sum_{j=0}^{\infty}\left(c\,2^{j(s_2-s_1)+(1+j)n(1/p_1-1/p_2)}\right)^{q_1q_2/(q_1-q_2)}\right)^{(q_1-q_2)/q_1q_2} \cdot \\ \qquad \cdot \left(\sum_{j=0}^{\infty} 2^{js_1q_1}\|(\varphi_j f^\wedge)^\vee\|_{p_1}^{q_1}\right)^{1/q_1} & \text{if } q_1>q_2 \\ \left(\sum_{j=0}^{\infty}\left(c\,2^{j(s_2-s_1)+(1+j)n(1/p_1-1/p_2)}\right)^{q_1} 2^{js_1q_1}\|(\varphi_j f^\wedge)^\vee\|_{p_1}^{q_1}\right)^{1/q_1} & \text{if } q_1\leq q_2 \end{cases}$$

$$\leq \begin{cases} c\,2^{n(1/p_1-1/p_2)}\left(\sum_{j=0}^{\infty} 2^{-j\delta q_1q_2/(q_1-q_2)}\right)^{(q_1-q_2)/q_1q_2} \cdot \\ \qquad \cdot \|f\,|\,B^{s_1}_{p_1q_1}(\mathbb{R}^n)\| & \text{if } q_1>q_2 \\ c\,2^{n(1/p_1-1/p_2)}\left(\sum_{j=0}^{\infty} 2^{-j\delta q_1}\, 2^{js_1q_1}\|(\varphi_j f^\wedge)^\vee\|_{p_1}^{q_1}\right)^{1/q_1} & \text{if } q_1\leq q_2 \end{cases}$$

$$\leq c_1\,\|f\,|\,B^{s_1}_{p_1q_1}(\mathbb{R}^n)\| \qquad \text{say}$$

(obvious modifications if $q_1=\infty$ or $q_2=\infty$).

Then, using an appropriate extension operator (see I.3.9, where we can choose a suitable χ and rename χE, with a smaller target space as detailed below, by E again), we can write $I(\Omega)$ as the composition

$$B^{s_1}_{p_1q_1}(\Omega) \xrightarrow{E} \{f\in B^{s_1}_{p_1q_1}(\mathbb{R}^n) : \operatorname{supp} f \subset B^n_\infty\} \xrightarrow{I} B^{s_2}_{p_2q_2}(\mathbb{R}^n) \xrightarrow{R} B^{s_2}_{p_2q_2}(\Omega) \qquad (4.8)$$

where $Ef|_\Omega = f$, R is the restriction operator, that is, $Rf = f|_\Omega$, and I is a natural embedding.

Formulae (I.3.7) and (I.3.9) permit us to define for each $N\in\mathbb{N}$,

$$F^N, F_{N,2} : \{f\in B^{s_1}_{p_1q_1}(\mathbb{R}^n) : \operatorname{supp} f \subset B^n_\infty\} \longrightarrow B^{s_2}_{p_2q_2}(\mathbb{R}^n) \qquad (4.9a)$$

respectively by

$$F^N f = f^N , \qquad F_{N,2} f = f_{N,2} , \qquad (4.9b)$$

with f^N and $f_{N,2}$ given respectively by (I.3.6) and (I.3.10) — where we suppose that the ψ of Lemma I.3.13 satisfies (I.3.8); formulae (I.3.7) and (I.3.9) together with (I.3.6) and (I.3.10) also permit us to say that F^N and $F_{N,2}$ above are operators. Hence the same is true for

$$F_{N,1} := I - F_{N,2} - F^N$$

and we can write $I(\Omega) = R\,I\,E = R(F_{N,1}+F_{N,2}+F^N)\,E = F_{N,1}(\Omega)+F_{N,2}(\Omega)+F^N(\Omega)$ if we define $F_{N,1}(\Omega) := R\,F_{N,1}\,E$ and similarly for the others. More precisely, we can write for each $N\in\mathbb{N}$,

$$I^{111}_{222}(\Omega) = F^{111}_{222,N,1}(\Omega) + F^{111}_{222,N,2}(\Omega) + F^{111,N}_{222}(\Omega) \tag{4.10}$$

if $p_1 \le p_2$.

4.4. For $p_1>p_2$ we define the operators F by

$$F^{111}_{222,N,1}(\Omega) := I^{212}_{222}(\Omega)\, F^{111}_{212,N,1}(\Omega)\ , \qquad N\in\mathbb{N}\,,$$

and similarly for the others (note that by I.3.7(iii) the $I(\Omega)$ that appears above is a continuous embedding). Thus, using (4.10) with $p_1=p_2$,

$$F^{111}_{222,N,1}(\Omega) + F^{111}_{222,N,2}(\Omega) + F^{111,N}_{222}(\Omega) = I^{212}_{222}(\Omega)\, I^{111}_{212}(\Omega) = I^{111}_{222}(\Omega)$$

and therefore (4.10) is also valid for $p_1>p_2$.

4.5. LEMMA. *For each* $N\in\mathbb{N}$

(i) $\|F^N(\Omega)\| \le c_1\, 2^{-N\delta}$ *and*

(ii) $\|F_{N,2}(\Omega)\| \le c_2\, 2^{-N\delta}$,

where c_1 *and* c_2 *are constants independent of* N.

Proof. (i) If $p_1\le p_2$ we have $\|F^N(\Omega)\| \le \|R\|\,\|F^N\|\,\|E\| \le c_1\, 2^{-N\delta}$, using (I.3.7) and (4.9).

If $p_1>p_2$ we apply the previous result with $p_1=p_2$, obtaining $\|F^{111,N}_{222}(\Omega)\| \le \|I^{212}_{222}(\Omega)\|\,\|F^{111,N}_{212}(\Omega)\| \le c_1\, 2^{-N(s_1-s_2)} = c_1\, 2^{-N\delta}$.

(ii) has a similar proof.

4.6. Consider again the case $p_1\le p_2$.

Let $N,H\in\mathbb{N}$ be such that $N>H>0$ and $2^{H+2}\sqrt{n} \ge N^2\delta^2$.

For each $j\in\{H,\dots,N\}$ define F_j by means of the following diagram

$$\begin{array}{ccc} \{ f\in B^{s_1}_{p_1q_1}(\mathbb{R}^n) : \operatorname{supp} f \subset B^n_\infty \} & \xrightarrow{\quad S_j \quad} & l^{M_j}_{p_1} \\ F_j \Big\downarrow & & \Big\downarrow Id_j \\ B^{s_2}_{p_2q_2}(\mathbb{R}^n) & \xleftarrow{\quad T_j \quad} & l^{M_j}_{p_2} \end{array} \qquad (4.11)$$

where M_j, S_j and T_j are as in Definition I.3.17 except that for S_j we have now s_1,p_1,q_1 instead of s,p,q respectively, and for T_j we have now s_2,p_2,q_2 instead of s,p,q respectively; Id_j is the same as $I^{p_1}_{p_2}(M_j)$, namely the natural embedding (it shall be clear from the context which p_1 and p_2 are being considered).

Thus we have $F_j = T_j\, Id_j\, S_j$. Defining $F_{H-1} := F_{N,1} - \sum_{j=H}^{N} F_j = I - F_{N,2} - F^N - \sum_{j=H}^{N} F_j$ and using the representation formula (I.3.4) together with the hypothesis $2^{H+2}\sqrt{n} \geq N^2\delta^2$ above, we obtain (recalling (I.3.5), (I.3.10), (I.3.12b) and (I.3.13b))

$$F_{H-1}f = C.\psi.\Big(\sum_{|m|_2\le c(N,0)} (\varphi_0\hat{f})^\vee(m)\, \check{\psi}(2\cdot -2m) + \sum_{j=1}^{H-1} \sum_{|m|_2\le c(N,j)} (\varphi_j\hat{f})^\vee(2^{-j}m)\, (\psi-\psi_\lambda)^\vee(2^{j+1}\cdot -2m) \Big), \qquad (4.12)$$

where $c(N,j) = \max\{N^2\delta^2, 2^{j+2}\sqrt{n}\}$ and $\sum_{j=1}^{H-1} \ldots = 0$ if $H=1$.

Define now $F_j(\Omega) := R F_j E$ and $F_{H-1}(\Omega) := R F_{H-1} E$, so that $I(\Omega)$ can be written as

$$I^{111}_{222}(\Omega) = F^{111}_{222,H-1}(\Omega) + \sum_{j=H}^{N} F^{111}_{222,j}(\Omega) + F^{111}_{222,N,2}(\Omega) + F^{111,N}_{222}(\Omega) \qquad (4.13)$$

if $p_1 \le p_2$.

4.7. Consider now the case $p_1 > p_2$.

Let $N,H\in\mathbb{N}$ be such that $N>H>0$ and $2^{H+2}\sqrt{n} \geq N^2\delta^2$.

As before we use the diagram (4.11) to define $F^{111}_{222,j}$, $j\in\{H,\ldots,N\}$, and note that for $F^{111}_{222,j}(\Omega) := R_{222}\, F^{111}_{222,j}\, E_{111}$ we have

$$F^{111}_{222,j}(\Omega) = I^{212}_{222}(\Omega)\, F^{111}_{212,j}(\Omega)\,. \qquad (4.14)$$

In fact, to justify this it suffices to say that each $f\in B^{s_1}_{p_1q_1}(\Omega)$ has the same image under both operators.

Defining now

$$F^{111}_{222,H-1}(\Omega) := F^{111}_{222,N,1}(\Omega) - \sum_{j=H}^{N} F^{111}_{222,j}(\Omega) \tag{4.15}$$

we have that

$$F^{111}_{222,H-1}(\Omega) = I^{212}_{222}(\Omega)\, R_{212}\, F^{111}_{212,H-1}\, E_{111}\ , \tag{4.16}$$

where $F^{111}_{212,H-1}f$ is given by (4.12), since by hypothesis $2^{H+2}\sqrt{n} \geq N^2\delta^2$ (remember that in this case $\delta = s_1 - s_2$).

Using (4.10) — which we proved is true for $p_1 > p_2$ in 4.4 — and (4.15) we also obtain that (4.13) is true in the case $p_1 > p_2$ too.

4.8. PROPOSITION. *Let* $\rho = \min\{1, p_2, q_2\}$ *and* $N, H \in \mathbb{N}$ *be such that* $N > H > 0$ *and* $2^{H+2}\sqrt{n} \geq N^2\delta^2$. *Let* $r \in \mathbb{N}$ *and consider* $r_{H-1}, r_H, \ldots, r_N \in \mathbb{N}$ *with* $\sigma := \sum_{j=H-1}^{N} r_j \leq r$. *Assume that* $\mathbf{s}$ *is any s-function. Then there is a constant* c *independent of* $N, H, r, r_{H-1}, r_H, \ldots, r_N$ *such that*

$$(s_r(I(\Omega)))^\rho \leq c\left(2^{-N\delta\rho} + (s_{r_{H-1}}(F_{H-1}))^\rho + \sum_{j=H}^{N} 2^{-j(s_1 - s_2 - n(1/p_1 - 1/p_2))\rho}(s_{r_j}(Id_j))^\rho\right), \tag{4.17}$$

where F_{H-1} *is given by (4.12) — but see also (4.19) below — and* Id_j *is as in diagram (4.11).*

Proof. Using (4.13) — which we proved is true for $p_1 > p_2$ in 4.7 —, Definition I.2.2, Proposition I.2.3 and Lemmas I.3.10 and 4.5, we obtain

$$\begin{aligned}
(s_r(I(\Omega)))^\rho &\leq \left(s_r\left(F_{H-1}(\Omega) + \sum_{j=H}^{N} F_j(\Omega)\right)\right)^\rho + \|F_{N,2}(\Omega)\|^\rho + \|F^N(\Omega)\|^\rho \\
&\leq \left(s_{\sigma-(N-H+1)}\left(F_{H-1}(\Omega) + \sum_{j=H}^{N} F_j(\Omega)\right)\right)^\rho + c_1 2^{-N\delta\rho} + c_2 2^{-N\delta\rho} \qquad (4.18)\\
&\leq (s_{r_{H-1}}(F_{H-1}(\Omega)))^\rho + \sum_{j=H}^{N}(s_{r_j}(F_j(\Omega)))^\rho + (c_1 + c_2)2^{-N\delta\rho}\ .
\end{aligned}$$

Using the definition of $F_{H-1}(\Omega)$ in the case $p_1 \leq p_2$ and (4.16) in the case $p_1 > p_2$ we can write

$$s_{r_{H-1}}(F^{111}_{222,H-1}(\Omega)) \le c_3 \begin{cases} s_{r_{H-1}}(F^{111}_{222,H-1}) & \text{if } p_1 \le p_2 \\ s_{r_{H-1}}(F^{111}_{212,H-1}) & \text{if } p_1 > p_2 \end{cases} = c_3 \cdot s_{r_{H-1}}(F_{H-1}), \qquad (4.19)$$

with F_{H-1} given by (4.12) and c_3 independent of H and r_{H-1} (and with the implicit understanding that when $p_1 > p_2$ the target space of F_{H-1} is $B^{s_2}_{p_1 q_2}(\mathbb{R}^n)$).

Using the definitions of $F_j(\Omega)$, $j \in \{H, \dots, N\}$, and formulae (I.3.14) we can also write

$$s_{r_j}(F_j(\Omega)) \le c_4\, 2^{-j(s_1-s_2-n(1/p_1-1/p_2))}\, s_{r_j}(Id_j)\,, \qquad j \in \{H, \dots, N\}, \qquad (4.20)$$

where c_4 is independent of r_j, $j \in \{H, \dots, N\}$, and $j \in \mathbb{N}$ and therefore it is independent of H and N.

Finally, using (4.19) and (4.20) in (4.18) we obtain (4.17).

4.9. **THEOREM.** *Given* $k \in \mathbb{N}$

$$x_k(I(\Omega)) \le c \cdot \begin{cases} k^{-(s_1-s_2)/n}, & 0<p_1,p_2\le 2 \\ k^{-((s_1-s_2)/n+1/p_2-1/2)}, & 0<p_1\le 2\le p_2\le\infty \\ k^{-((s_1-s_2)/n+1/2-1/p_1)}, & 0<p_2\le 2<p_1\le\infty \text{ and } s_1-s_2>n/p_1 \\ k^{-((s_1-s_2)/n+1/p_2-1/p_1)}, & 2\le p_1\le p_2\le\infty \\ & \text{or } 2\le p_2<p_1\le\infty \text{ and } s_1-s_2>n\dfrac{1/p_2-1/p_1}{p_1/2-1} \\ k^{-(s_1-s_2)p_1/(2n)}, & 0<p_2\le 2<p_1<\infty \text{ and } s_1-s_2<n/p_1 \\ & \text{or } 2\le p_2<p_1<\infty \text{ and } s_1-s_2<n\dfrac{1/p_2-1/p_1}{p_1/2-1} \end{cases} \qquad (4.21)$$

where c *is independent of* k *(and the usual rules of calculation on* $\bar{\mathbb{R}}$ *apply).*

Proof. In the sequel, whenever we refer to *lines* we mean those on the left-hand side within the above brace, some of them corresponding to two lines — which we shall call *cases* — on the right-hand side.

Using Proposition I.2.6(a) and (I.3.11) we obtain immediately the result for the first case of the fourth line. Also the first line for $0<p_2<p_1\le 2$ is easy to obtain because $I^{111}_{222}(\Omega) = I^{212}_{222}(\Omega)\, I^{111}_{212}(\Omega)$ — recall Lemma I.3.7(ii),(iii) — and therefore

$$x_k(I^{111}_{222}(\Omega)) \le c_1\, x_k(I^{111}_{212}(\Omega)) \le c_1\, a_k(I^{111}_{212}(\Omega)) \le c\, k^{-(s_1-s_2)/n},$$

using again Proposition I.2.6(a) and (I.3.11).

Now let $k\in\mathbb{N}$ be large enough so that

$$N := \left[\tfrac{\alpha}{n}\log_2 k\right],\ L := \left[\tfrac{1}{n}\log_2 k\right],\ H := \left[\log_2\frac{N^2\delta^2}{\sqrt{n}} - 1\right] \in \mathbb{N},$$

where α is to be fixed later on and for the moment we only restrict it to be greater or equal to 1 and independent of k.

Note that $2^{L+2}\sqrt{n} \ge 2.2^{N/\alpha}\sqrt{n} > 2\,N^2\delta^2 \ge 2^{H+2}\sqrt{n}$ for N large enough, or for k large enough, as we are assuming.

Hence $N \ge L > H > 0$.

From the definition of H it follows easily that $2^{H+2}\sqrt{n} \ge N^2\delta^2$, and thus N and H satisfy the hypotheses of Proposition 4.8.

Since F_{H-1} given by (4.12) has finite rank we can consider

$$r_{H-1} = \operatorname{rank} F_{H-1} + 1 \tag{4.22}$$

and thus

$$s_{r_{H-1}}(F_{H-1}) = 0. \tag{4.23}$$

Also Id_j, $j\in\{H,\dots,N\}$, given by (4.11) have finite rank M_j, so we can consider

$$r_j = M_j + 1 \qquad \text{for} \quad j\in\{H,\dots,L-1\} \tag{4.24}$$

and thus

$$s_{r_j}(Id_j) = 0 \qquad \text{for} \quad j\in\{H,\dots,L-1\}. \tag{4.25}$$

To prove (4.21) for the cases which were not considered in the very beginning of this proof, we are going to use

$$r_j = [k^{1+\beta/n}\, 2^{-j\beta}], \qquad L\le j\le N, \tag{4.26}$$

except in the last line of (4.21), where we use instead

$$r_j = [k^{1-\alpha\beta/n}\, 2^{j\beta}], \qquad L\le j\le N, \tag{4.27}$$

where $\beta>0$ is to be fixed later on independently of the value of k (note that r_j in (4.26) and (4.27) belong to $\mathbb{N}$ for k large enough if β is small enough — we are going to see that we can choose β in this way).

From (4.12) and the fact that for $0\le j<H$ we have $2^{j+2}\sqrt{n} \le N^2\delta^2$ (see the definition of H) it follows that we can estimate $\operatorname{rank} F_{H-1}$ from above by $c_1 H N^{2n}$ with c_1 independent of H and N and therefore of k. For $H\le j<L$ we can estimate M_j (the number of $m\in\mathbb{Z}^n$ such that $|m|_2\le 2^{j+2}\sqrt{n}$) from above by $c_2 2^{jn}$ with c_2 independent of $j\in\mathbb{N}$. Consequently we have , using (4.22) and (4.24),

$$\begin{aligned}\sum_{j=H-1}^{L-1} r_j &= r_{H-1} + \sum_{j=H}^{L-1} r_j = \operatorname{rank} F_{H-1} + 1 + \sum_{j=H}^{L-1} (M_j+1)\\ &\le c_1 H N^{2n} + 1 + \sum_{j=H}^{L-1} (c_2 2^{jn} + 1) \le c_1 N^{2n+1} + 1 + c_2 \frac{1-2^{nL}}{1-2^n} + L\\ &\le \left(c_1 \frac{N^{2n+1}}{k} + \frac{1}{k} + \frac{c_2}{2^n-1}\frac{2^{nL}}{k} + \frac{L}{k}\right) k \le c_3 k,\end{aligned} \tag{4.28}$$

where c_3 is independent of k.

For r_j given by (4.26) we have

$$\sum_{j=L}^{N} r_j \le \sum_{j=L}^{N} k^{1+\beta/n} 2^{-j\beta} \le k^{1+\beta/n} \frac{2^{-L\beta}}{1-2^{-\beta}} \le \frac{2^\beta}{1-2^{-\beta}} k \tag{4.29}$$

and for r_j given by (4.27) we have

$$\sum_{j=L}^{N} r_j \le \sum_{j=L}^{N} k^{1-\alpha\beta/n} 2^{j\beta} \le k^{1-\alpha\beta/n} 2^\beta \frac{1-2^{N\beta}}{1-2^\beta} \le \frac{2^\beta}{2^\beta-1} k\,. \tag{4.30}$$

Now (4.28) to (4.30) imply that

$$\sum_{j=H-1}^{N} r_j \le c_4 k,$$

where c_4 is independent of k, and therefore we can finally choose $r = c_4 k$ and apply Proposition 4.8 with $\mathbf{s}$ the s-function defined by the Weyl numbers and obtain, taking (4.23) and (4.25) into account,

$$(x_{c_4 k}(I(\Omega)))^\rho \leq c \left(2^{-N\delta\rho} + \sum_{j=L}^{N} 2^{-j(s_1-s_2-n(1/p_1-1/p_2))\rho}(x_{r_j}(I_{p_2}^{p_1}(M_j))^\rho)\right), \qquad (4.31)$$

with c independent of k.

Next we use Propositions 3.5, 3.7, Remark 3.6 and the upper bound $c_2 2^{jn}$ for M_j to estimate the right-hand side of (4.31), thus obtaining the required upper estimates for $x_{c_4 k}(I(\Omega))$ — or, which is the same, for $x_k(I(\Omega))$ — in the four cases highlighted below. Since the proof of these from now on is classical (apart from the harmless extra exponent ρ, which we get rid of in the end) we shall omit the details and refer to [König 1986, p.188], where some are worked out in a similar context. Here we just list the values of α and β that are to be chosen in each of the following cases of (4.21):

— in the case $0<p_1\leq p_2\leq 2$ of the first line we choose $\alpha=(s_1-s_2)/\delta$ (which is ≥ 1) and $\beta>0$ such that $\delta-\beta(1/p_1-1/p_2)>0$;

— for the second line we choose $\alpha=(s_1-s_2)/\delta$ (which is ≥ 1) and $\beta>0$ such that $\delta-\beta(1/p_1-1/2)>0$;

— for the second case of the fourth line we choose $\alpha=\dfrac{\delta+n(1/p_2-1/p_1)}{\delta}$ (which is ≥ 1) and $\beta>0$ such that $\delta-(1/p_2-1/p_1)\left(\dfrac{n}{p_1/2-1}+\dfrac{\beta}{1-2/p_1}\right)>0$ (remember that here and in the remaining cases $\delta=s_1-s_2$);

— for the second case of the fifth line we choose $\alpha=p_1/2$ (which is ≥ 1) and $\beta>0$ such that $(1/p_2-1/p_1)\left(\dfrac{n}{p_1/2-1}-\dfrac{\beta}{1-2/p_1}\right)-\delta>0$.

Note that each $\beta>0$ considered above can be chosen as small as we want.

The two last cases and a reasoning intrinsic to the theory of function spaces give the result for the remaining cases, namely the one corresponding to the third line and the first case of the fifth line. We exemplify with the latter:

Using Lemma I.3.7(ii),(iii) we can write $I_{s_2 p_2 q_2}^{s_1 p_1 q_1}(\Omega) = I_{s_2 p_2 q_2}^{s_2\, 2\, q_2}(\Omega)\, I_{s_2\, 2\, q_2}^{s_1 p_1 q_1}(\Omega)$,

and

$$x_k(I^{s_1 p_1 q_1}_{s_2 p_2 q_2}(\Omega)) \le c_5\, x_k(I^{s_1 p_1 q_1}_{s_2\, 2\, q_2}(\Omega)) \le c_6\, k^{-(s_1-s_2)p_1/(2n)}$$

if $s_1-s_2 < n\dfrac{1/2-1/p_1}{p_1/2-1}$, that is, if $s_1-s_2 < n/p_1$.

4.10. *Remark.* In the case $0<\max\{p_2,2\}<p_1<\infty$ and $s_1-s_2 = n\dfrac{1/\max\{p_2,2\}-1/p_1}{p_1/2-1}$, using $r_j = [k/\log_2 k]$ for $L\le j\le N$, the best we can prove with the same methods of the preceding theorem is that

$$x_k(I(\Omega)) \le c\, k^{-(s_1-s_2)p_1/(2n)}\, (\log k)^{(s_1-s_2)p_1/(2n)+1/\min\{1,q_2\}}.$$

Note that in this case $\dfrac{(s_1-s_2)p_1}{2n} = \dfrac{s_1-s_2}{n} + \dfrac{1}{\max\{p_2,2\}} - \dfrac{1}{p_1}$.

4.11. We now turn to the lower estimates for $x_k(I(\Omega))$.

From Lemma I.3.19 we easily deduce

$$x_k(I^{p_1}_{p_2}(2^{jn})) \le \|B\|\, x_k(I(\Omega))\, \|A\| \le c_1 c_2\, 2^{j(s_1-n/p_1)}\, 2^{-j(s_2-n/p_2)}\, x_k(I(\Omega)),$$

from which follows that

$$x_k(I(\Omega)) \ge c\, 2^{-j(s_1-s_2-n/p_1+n/p_2)}\, x_k(I^{p_1}_{p_2}(2^{jn})), \tag{4.32}$$

where $c>0$ is independent of $j,k\in\mathbb{N}$.

4.12. THEOREM. *Given* $k\in\mathbb{N}$

$$x_k(I(\Omega)) \ge c\cdot \begin{cases} k^{-(s_1-s_2)/n}, & 0<p_1,p_2\le 2 \\ k^{-((s_1-s_2)/n+1/p_2-1/2)}, & 0<p_1\le 2\le p_2\le\infty \\ k^{-((s_1-s_2)/n+1/2-1/p_1)}, & 0<p_2\le 2\le p_1\le\infty \\ k^{-((s_1-s_2)/n+1/p_2-1/p_1)}, & 2\le p_1,p_2\le\infty \\ k^{-(s_1-s_2)p_1/(2n)}, & 0<\max\{p_2,2\}\le p_1<\infty \end{cases} \tag{4.33}$$

where $c>0$ *is independent of* k.

Proof. We use Propositions 3.5 and 3.8 together with (4.32), where for each k we make a convenient choice of j. We point out this choice next, skipping over the details of routine calculations:

– for the first four lines of (4.33) choose $j=\left[\frac{1}{n}\log_2(2k)+1\right]$ (indeed, the only non-trivial situation here is when $2<p_2<p_1\le\infty$, which is dealt with below);

– for the last line of (4.33) choose $j=\left[\frac{p_1}{2n}\log_2 k+1\right]$ to deal with the situation $0<p_2\le 2<p_1<\infty$, and for the case $2<p_2\le p_1<\infty$ use the fact that $I^{s_1 p_1 q_1}_{s_2\, 2\, q_2}(\Omega) = I^{s_2 p_2 q_2}_{s_2\, 2\, q_2}(\Omega)\, I^{s_1 p_1 q_1}_{s_2 p_2 q_2}(\Omega)$ – recall Lemma I.3.7(iii) – in order to obtain the desired estimate by reduction to the previous situation.

As mentioned above, we consider now the case $2<p_2<p_1\le\infty$:

For $\varepsilon>n/p_2$ write (because of Lemma I.3.7(ii))

$$I^{s_1\ p_1 q_1}_{s_2-\varepsilon\ 2\ q_2}(\Omega) = I^{s_2\ p_2 q_2}_{s_2-\varepsilon\ 2\ q_2}(\Omega)\ I^{s_1 p_1 q_1}_{s_2 p_2 q_2}(\Omega)$$

and, using the multiplicativity of the Weyl numbers (Proposition I.2.7), the third line of (4.33) and the third line of (4.21),

$$\begin{aligned} c_1\,(2k)^{-((s_1-s_2+\varepsilon)/n+1/2-1/p_1)} &\le x_{2k}(I^{s_1\ p_1 q_1}_{s_2-\varepsilon\ 2\ q_2}(\Omega)) \\ &\le x_k(I^{s_2\ p_2 q_2}_{s_2-\varepsilon\ 2\ q_2}(\Omega))\ x_k(I^{s_1 p_1 q_1}_{s_2 p_2 q_2}(\Omega)) \\ &\le c_2\, k^{-((s_2-s_2+\varepsilon)/n+1/2-1/p_2)}\ x_k(I^{s_1 p_1 q_1}_{s_2 p_2 q_2}(\Omega))\,, \end{aligned}$$

that is,

$$x_k(I^{s_1 p_1 q_1}_{s_2 p_2 q_2}(\Omega)) \ge c\, k^{-((s_1-s_2)/n+1/p_2-1/p_1)}\,,$$

as required.

4.13. COROLLARY. *The estimates* k^u *in (4.21) hold as two-sided estimates for the Weyl numbers of* $I(\Omega)$, *that is,*

$$x_k(I(\Omega)) \approx k^u \qquad \text{as} \quad k\to\infty\ .$$

Proof. This is a direct consequence of Theorems 4.9 and 4.12.

4.14. COROLLARY. *Remark 4.10, Theorem 4.12 and Corollary 4.13 remain valid if instead of* $I(\Omega)$ *we have any one of the following natural embeddings (also with the restrictions on the parameters set forth in subsection 4.1, of course):*

$$F^{s_1}_{p_1q_1}(\Omega) \longrightarrow B^{s_2}_{p_2q_2}(\Omega) \qquad \text{if} \quad p_1<\infty$$

$$B^{s_1}_{p_1q_1}(\Omega) \longrightarrow F^{s_2}_{p_2q_2}(\Omega) \qquad \text{if} \quad p_2<\infty$$

$$F^{s_1}_{p_1q_1}(\Omega) \longrightarrow F^{s_2}_{p_2q_2}(\Omega) \qquad \text{if} \quad p_1,p_2<\infty\,.$$

Proof. This is a straightforward consequence of Lemma I.3.7(i),(ii), the defining properties of s-functions and the results mentioned in the statement of this corollary.

CHAPTER III

EIGENVALUES OF THE DIRICHLET LAPLACIAN FOR SOME 2-DIMENSIONAL DOMAINS

1. INTRODUCTION

1.1. In II.1 we mentioned in passing that there is another reason, besides the one explained there, why the s-numbers should be of interest. We dwell upon it now.

We have pointed out in Remark I.4.12 that for the variational triplet (V,H,b) with V densely and compactly embedded in H,

$$d_j(S_{b+\lambda_0},H) = (\lambda_{j+1}+\lambda_0)^{-1/2}, \qquad j\in\mathbb{N}_0, \tag{1.1}$$

where λ_0 is a real constant such that $b+\lambda_0$ is strongly coercive on V (please recall the relevant definitions in section I.4). It is apparent that $d_j(S_{b+\lambda_0},H)$ resembles a Kolmogorov number and if, for example, we consider $H=L_2(\Omega)$, for some non-void bounded open set $\Omega\subset\mathbb{R}^n$, and $b=(\nabla\cdot,\nabla\cdot)_\Omega$ the form on $V=H_0^1(\Omega)$ — see, in particular, I.4.16 — then we can take $\lambda_0=1$ and obtain, from (1.1),

$$s_j(I) = (\lambda_j+1)^{-1/2}, \qquad j\in\mathbb{N}, \tag{1.2}$$

where $s_j(I)$ denotes the j-th s-number of the natural embedding $I: V\longrightarrow H$. This explicit formula permits thus the knowledge of the eigenvalues of the differential operator under consideration (in this case the Dirichlet Laplacian — see I.4.15) directly from the knowledge of the s-numbers of an embedding operator.

For results on the asymptotic distribution of eigenvalues of differential operators using an interplay with s-numbers such as (1.2), or even (1.1), see [Birman/Solomjak 1980], [Métivier 1977], [Triebel 1978, 5.6, 6.6, 7.8] and references given there. See also [Evans/Harris 1989] for a recent example involving the Neumann Laplacian for a trumpet-shaped domain Ω.

1.2. In this chapter we shall be concerned with the variational triplet considered above, which, as pointed out in I.4.15, determines the well-known Dirichlet Laplacian. Some information about the asymptotic behaviour of the corresponding eigenvalues could be derived from Theorem II.4.9, by specialization, but this would be to reverse the natural order of things — much more is already known in this case.

Before we present any results we would like to set the framework for this chapter and modify slightly some notation used till now. Since our main aim here is to present examples of situations that can occur, we shall restrict our attention to *non-void bounded open subsets* B *of* $\mathbb{R}^2$. *In the sequel* B *will therefore stand for such a set.* The only norms to be used on $\mathbb{R}$ and $\mathbb{R}^2$ are the Euclidean ones, so that we shall denote them simply by $|\cdot|$. By $\mathrm{Vol}(\cdot)$ we shall mean Lebesgue measure in $\mathbb{R}^2$ (that is, what has been represented by $\mathrm{Vol}_2(\cdot)$), though in integrals we shall use the familiar dx (possibly with another letter in place of x). We shall also use the shorthand

$$N(\lambda) := N^B(\lambda) := N(\lambda, H_0^1(B), L_2(B), (\nabla\cdot,\nabla\cdot)_B) \tag{1.3}$$

for the counting function associated with the Dirichlet Laplacian for B (recall Proposition I.4.11 for the connection between N^B and the eigenvalues of the operator in question, which explains the name counting function). Another function also commonly used to characterize the distribution of the eigenvalues of $-\Delta_{\partial B}^B$ is the *partition function*

$$Z(t) := Z^B(t) := \int_0^\infty e^{-\lambda t}\, dN^B(\lambda) = \sum_{j=1}^\infty e^{-\lambda_j t}, \qquad t>0, \tag{1.4}$$

where the integral is taken in the sense of Lebesgue-Stieltjes.

The following is known to hold [Lapidus a, Th. 2.3]:

1.3. **Proposition.** *Let* D^B *be the Minkowski dimension of* ∂B *relative to* B *(see Definition 3.2 below). If* $D^B \in (1,2]$ *(that is,* ∂B *is* fractal*), then for all* $d > D^B$,

$$N^B(\lambda) = \frac{\mathrm{Vol}(B)}{4\pi}\lambda + O(\lambda^{d/2}) \qquad \text{as} \quad \lambda \to \infty \tag{1.5}$$

and

$$Z^B(t) = \frac{\mathrm{Vol}(B)}{4\pi} t^{-1} + O(t^{-d/2}) \qquad \text{as} \quad t \to 0^+ . \tag{1.6}$$

If, furthermore, the D^B*-dimensional upper Minkowski content of* ∂B *relative to* B *(see Definition 3.2) is finite, then the above estimates also hold with* $d = D^B$.

1.4. It is in the sense of (1.5) or (1.6) (and refinements of these) that the meaning of an expression such as *the asymptotic distribution of the eigenvalues of the Dirichlet Laplacian* is interpreted here (of course, from (1.5) we can obtain more direct assertions for the eigenvalues — see [Caetano b, proof of Prop. 4.1]). Also, we shall refer to the *asymptotic expansion for* N^B as a possible expansion $N^B(\lambda) = \frac{\mathrm{Vol}(B)}{4\pi}\lambda + c_1 g(\lambda) + \ldots$ (with g a function and c_1 a constant) and to $c_1 g(\lambda)$ as the *possible second term*, or even only *second term*, in that expansion, even if it doesn't exist. Thus, if for example we can prove that

$$\frac{\mathrm{Vol}(B)}{4\pi}\lambda - N^B(\lambda) \approx g(\lambda) \qquad \text{as} \quad \lambda \to \infty ,$$

we shall say that *the second term* (or *the remainder*) *in the asymptotic expansion* (or *asymptotic formula*) *for* N^B *is of the form of the function* g, or something equivalent. Analogously for Z^B.

For some remarks on the history of this subject and also for results in this and other contexts, see [Métivier 1977], [Lapidus a] and references therein. See also [Fleckinger/Vasil'ev a] and [Lapidus/Pomerance a].

In particular it is well-known that the first term in the asymptotic expansion for $N^B(\lambda)$ is always $\frac{\text{Vol}(B)}{4\pi}\lambda$ (remember that B is allowed to be any non-void bounded open subset of $\mathbb{R}^2$), that is,

$$\lim_{\lambda\to\infty} \frac{N^B(\lambda)}{\lambda} = \frac{\text{Vol}(B)}{4\pi} .$$

However, it is also known that the second term can have various forms, and the forms it can exhibit have been connected with the *behaviour* of the boundary of B. This is the philosophy behind Proposition 1.3 above, and it is indeed possible to give examples where the second term in the asymptotic expansion for N^B has the form $\lambda^{D/2}$ (see [Lapidus a, Ex. 5.1'], for example, or Corollary 3.14 below in the case $h(\lambda)\equiv 1$).

1.5. Our main aim in this chapter is to investigate through examples (including ones in which B is connected) how different the forms of the second term can be. At the same time we want to compare our results with Proposition 1.3 in order to see to what extent there is sharpness over there.

Section 2 is devoted to the derivation of useful estimates when B is a rectangle and there can be mixed Dirichlet-Neumann boundary conditions. Results of this kind seem to be known by everybody, but we couldn't find any exact and general enough reference, let alone proofs, for them. Particular cases can however be found in [Reed/Simon 1978, pp. 264-267] or [Edmunds/Evans 1987, XI.2.3].

The geometrical aspect of the sets B considered in section 3 is inspired by Examples 5.1 and 5.1' of [Lapidus a]. Our novelty is that we decided to broaden the class of functions a to which the sets B are associated and, since we succeeded in overcoming the difficulty in finding workable expressions for inverses and composition of functions that arise naturally, we were led to some interesting results concerning the asymptotic expansion for N^B.

In section 4 we go a step further by considering connected sets B obtained from the disconnected ones dealt with in section 3. We prove then that all the relevant previous results hold also in the new situation, using material from sections I.4, 2 and 3 through a Dirichlet-Neumann bracketing method. We also show how to carry some of these results to corresponding ones for the partition function.

Part of the contents of sections 3 and 4 is going to appear in [Caetano b]. There is, however, an essential difference: the results presented here are more precise. In [Caetano b] we used probabilistic methods in order to tackle the case of connected B (following a technique used in [Brossard/Carmona 1986]), studying thus first the partition function. The problem is that some information is lost when going then to the counting function. What made us change our approach was a paper by van den Berg [a] where a Dirichlet-Neumann bracketing technique is also used (though something in this direction could already have been seen in [Métivier 1977, VII]). Anyway, both van den Berg and Lapidus encouraged us to try this approach[1].

2. Dirichlet-Neumann Laplacians for rectangles

2.1. Throughout all this section our underlying set B will be a rectangle $R =]a_1,b_1[\times]a_2,b_2[$ in $\mathbb{R}^2$, where a_1,b_1,a_2,b_2 are given real numbers such that $a_1<b_1$, $a_2<b_2$. In addition, we set

(1) $S_1 := \partial R$;

(2) $S_2 := \partial R \setminus]a_1,b_1[\times\{b_2\}$;

(3) $S_3 := \partial R \setminus]a_1,b_1[\times\{a_2\}$.

Our aim is to determine the eigenvalues of any of the operators $-\Delta^{R}_{S_j}$, $j=1,2,3$.

[1] Once upon a time there was a very nice meeting, with a very nice weather, held in the very nice city of Cardiff... It was July 1990 and I would like to take the opportunity to thank Prof. W. D. Evans for the organization of that informal gathering of people addicted to eigenvalue problems.

2.2. If we assume for the moment that we are dealing with the corresponding classical operator $-\Delta := -D_1^2 - D_2^2$ with Dirichlet boundary conditions on S_j and Neumann boundary conditions on $\partial R \setminus S_j$, $j=1,2,3$, we can guess, by separation of variables, what the eigenvalues and the corresponding eigenvectors should be: we have respectively (with the eigenvectors normalized in $L_2(R)$)

(1) $\left(\frac{k\pi}{b_1-a_1}\right)^2 + \left(\frac{l\pi}{b_2-a_2}\right)^2$, $k,l \in \mathbb{N}$, are eigenvalues with corresponding eigenvectors $\left(\frac{4}{(b_1-a_1)(b_2-a_2)}\right)^{1/2} \sin\left(k\pi\frac{x_1-a_1}{b_1-a_1}\right) \sin\left(l\pi\frac{x_2-a_2}{b_2-a_2}\right)$;

(2) $\left(\frac{k\pi}{b_1-a_1}\right)^2 + \left(\frac{(l+1/2)\pi}{b_2-a_2}\right)^2$, $k \in \mathbb{N}$, $l \in \mathbb{N}_0$, are eigenvalues with corresponding eigenvectors $\left(\frac{4}{(b_1-a_1)(b_2-a_2)}\right)^{1/2} \sin\left(k\pi\frac{x_1-a_1}{b_1-a_1}\right) \cos\left((l+1/2)\pi\frac{x_2-b_2}{b_2-a_2}\right)$;

(3) $\left(\frac{k\pi}{b_1-a_1}\right)^2 + \left(\frac{(l+1/2)\pi}{b_2-a_2}\right)^2$, $k \in \mathbb{N}$, $l \in \mathbb{N}_0$, are eigenvalues with corresponding eigenvectors $\left(\frac{4}{(b_1-a_1)(b_2-a_2)}\right)^{1/2} \sin\left(k\pi\frac{x_1-a_1}{b_1-a_1}\right) \cos\left((l+1/2)\pi\frac{x_2-a_2}{b_2-a_2}\right)$.

From the way these functions — $g(x_1,x_2)$ say — and corresponding values — λ say — were obtained, we have that

$$-\Delta g = \lambda g \quad \text{and } g \text{ satisfies the prescribed boundary conditions} \tag{2.1}$$

(alternatively, we can verify this by direct computation with the functions and values given above).

2.3. Now we claim that for each case $j \in \{1,2,3\}$ the functions in 2.2(j) belong to $\mathcal{D}(-\Delta_{S_j}^R)$.

We deal only with $j=2$, the case $j=3$ being analogous and the case $j=1$ being even simpler.

Let us then denote, for (k,l) fixed in $\mathbb{N}\times\mathbb{N}_0$ and $x \equiv (x_1,x_2)$,

$$g_R(x) = \left(\frac{4}{(b_1-a_1)(b_2-a_2)}\right)^{1/2} \sin\left(k\pi\frac{x_1-a_1}{b_1-a_1}\right) \cos\left((l+1/2)\pi\frac{x_2-b_2}{b_2-a_2}\right)$$

with domain R, and let g_{2R} denote the function with the same expression but with domain $2R$, where $2R :=]a_1,b_1[\times]a_2,b_2+(b_2-a_2)[$. Note that $g_{2R}(x)=0$ for $x\in\partial 2R$, so that an easy application of the mean value theorem shows that

$$\frac{|g_{2R}(x)|}{\text{dist}(x,\partial 2R)} \le \sup_{y\in 2R} |\nabla g_{2R}(y)| \le c \qquad \text{for all } x\in 2R$$

where c depends only on R and (k,l). Hence $g_{2R}\in H^1(2R)$ and $\frac{g_{2R}}{\text{dist}(\cdot,\partial 2R)}\in L_2(2R)$, therefore — see [Edmunds/Evans 1987, p.223] — $g_{2R}\in H_0^1(2R)$.

Recall now that, as a consequence of Lemma I.4.14, $H_0^1(2R)=H_{\partial 2R}^1(2R)$, and consequently there exists a sequence $(g_n)_n\subset C_{\partial 2R}^\infty(\mathbb{R}^n)$ such that $g_n|_{2R}\xrightarrow[n\to\infty]{} g_{2R}$ in $H^1(2R)$. However, since $S_2\subset\partial 2R$ we have also $(g_n)_n\subset C_{S_2}^\infty(\mathbb{R}^n)$, and $\|g_n|_R-g_R\|_{H^1(R)} \le \|g_n|_{2R}-g_{2R}\|_{H^1(2R)} \xrightarrow[n\to\infty]{} 0$; that is, $g_R\in H_{S_2}^1(R)$.

Besides, given $\varphi\in C_{S_2}^\infty(R)$

$$\begin{aligned}(\nabla g_R,\nabla\varphi)_R &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} D_1 g_R(x)\overline{D_1\varphi(x)}\,dx_1\,dx_2 + \int_{a_1}^{b_1}\int_{a_2}^{b_2} D_2 g_R(x)\overline{D_2\varphi(x)}\,dx_2\,dx_1 \\ &= \int_{a_2}^{b_2}-\int_{a_1}^{b_1} D_1^2 g_R(x)\overline{\varphi(x)}\,dx_1\,dx_2 + \int_{a_1}^{b_1}-\int_{a_2}^{b_2} D_2^2 g_R(x)\overline{\varphi(x)}\,dx_2\,dx_1 = (-\Delta g_R,\varphi)_{L_2(R)},\end{aligned}$$

so that, by I.4.15 and Proposition I.4.8(iii), $g_R\in\mathcal{D}(-\Delta_{S_2}^R)$ — which proves our claim — and, moreover, $-\Delta_{S_2}^R g_R=-\Delta g_R$.

The last equality implies, in view of (2.1), that

$$-\Delta_{S_2}^R g_R = \left(\left(\frac{k\pi}{b_1-a_1}\right)^2+\left(\frac{(l+1/2)\pi}{b_2-a_2}\right)^2\right) g_R,$$

that is, assertion 2.2(2) is true for our operator $-\Delta_{S_2}^R$.

Analogously, assertions 2.2(1) and 2.2(3) hold also, respectively, for $-\Delta_{S_1}^R$ and $-\Delta_{S_3}^R$.

2.4. Now we show that for each $j\in\{1,2,3\}$ the functions in 2.2(j) form a complete orthonormal system in $L_2(R)$.

We use the well-known fact that

$$\{\sqrt{2/\pi}\,\sin(m\cdot)\}_{m\in\mathbb{N}} \tag{2.2}$$

and

$$\{1/\sqrt{\pi}, \sqrt{2/\pi}\cos(m\cdot)\}_{m\in\mathbb{N}} \tag{2.3}$$

are two complete orthonormal systems in $L_2(]0,\pi[)$ — see [Kadlec/Kufner 1971, 4.2.3] for example. Furthermore, it can then be easily seen that the system

$$\{\sqrt{2/\pi}\cos((m+1/2)\cdot)\}_{m\in\mathbb{N}_0} \tag{2.4}$$

is also complete orthonormal in $L_2(]0,\pi[)$. In fact, leaving aside the trivial part of showing that the orthonormality holds, the completeness can be proved as follows:

Assume that $g\in L_2(]0,\pi[)$ satisfies $(g, \sqrt{2/\pi}\cos((m+1/2)\cdot))_{L_2(]0,\pi[)} = 0$ for all $m\in\mathbb{N}_0$. For m=0 this implies

$$\int_0^\pi g(x).1/\sqrt{\pi}.\cos\frac{x}{2}\,dx = 0,$$

and for each $m\in\mathbb{N}$

$$\int_0^\pi g(x).\cos\frac{x}{2}.\sqrt{2/\pi}.\cos(mx)\,dx =$$

$$= \frac{1}{2}\int_0^\pi g(x).\sqrt{2/\pi}.(\cos((m+1/2)x) + \cos((m-1/2)x))\,dx = 0,$$

so that, by the completeness of (2.3), $g(x)\cos\frac{x}{2} = 0$ a.e. in $]0,\pi[$. Since $\cos\frac{x}{2}$ is never zero for $x\in]0,\pi[$, then $g(x)=0$ a.e. in $]0,\pi[$, that is, $g=0$ in $L_2(]0,\pi[)$, and this finishes the proof of the completeness of (2.4).

From (2.2) and (2.4) we get, by simple change of variables in integrals, the completeness of the following orthonormal systems in $L_2(]a,b[)$, for given $a,b\in\mathbb{R}$ $(a<b)$:

$$\left\{\sqrt{2/(b-a)}\,\sin\left(m\pi\frac{x-a}{b-a}\right)\right\}_{m\in\mathbb{N}},$$

$$\left\{\sqrt{2/(b-a)}\,\cos\left((m+1/2)\pi\frac{x-b}{b-a}\right)\right\}_{m\in\mathbb{N}_0},$$

$$\left\{\sqrt{2/(b-a)}\,\cos\left((m+1/2)\pi\frac{x-a}{b-a}\right)\right\}_{m\in\mathbb{N}_0}.$$

Now we use the fact that *if* $\{\varphi_k\}_k$, $\{\psi_l\}_l$ *are complete orthonormal in* $L_2(]a_1,b_1[)$, $L_2(]a_2,b_2[)$ *respectively, then* $\{\varphi_k\psi_l\}_{(k,l)}$ *is complete orthonormal in* $L_2(]a_1,b_1[\times]a_2,b_2[)$ – see [Kadlec/Kufner 1971, p. 122] – to conclude the completeness of the system of functions in 2.2(j) for each $j\in\{1,2,3\}$, as required.

2.5. Let us recall what has been achieved so far and derive some consequences which will accomplish our main goal in this section.

By 2.3 we know that all the values listed in 2.2(j) are eigenvalues of $-\Delta^{\mathbf{R}}_{S_j}$, $j\in\{1,2,3\}$, and by 2.4 we know that the repetition of eigenvalues performed in 2.2(j) does not exceed the repetition according to multiplicity. On the other hand, 2.4 and the self-adjointness of $-\Delta^{\mathbf{R}}_{S_j}$ imply that there can be no more eigenvalues (repeated according to multiplicity or different ones) of $-\Delta^{\mathbf{R}}_{S_j}$ apart from those, so that 2.2(j) *lists precisely the eigenvalues of* $-\Delta^{\mathbf{R}}_{S_j}$ *with repetition according to multiplicity.*

From this, and taking account of I.4.16, we can derive estimates for the associated counting functions, which will prove useful in the sequel: given $\lambda>0$

(i) $N^{\mathbf{R}}(\lambda) = N(\lambda,H^1_{S_1}(\mathbf{R}),L_2(\mathbf{R}),(\nabla\cdot,\nabla\cdot)_{\mathbf{R}}) = \#\{(k,l)\in\mathbb{N}^2 : \left(\frac{k\pi}{b_1-a_1}\right)^2+\left(\frac{l\pi}{b_2-a_2}\right)^2 \le \lambda\}$, the number of points $(k,l)\in\mathbb{N}^2$ inside or on the ellipse

$$\frac{k^2}{\lambda(b_1-a_1)^2/\pi^2}+\frac{l^2}{\lambda(b_2-a_2)^2/\pi^2} = 1 ,$$

from which follows

$$\frac{\mathrm{Vol}(\mathbf{R})}{4\pi}\lambda - \frac{b_1-a_1+b_2-a_2}{\pi}\sqrt{\lambda} \le N^{\mathbf{R}}(\lambda) \le \frac{\mathrm{Vol}(\mathbf{R})}{4\pi}\lambda \qquad (2.5)$$

and,

$$\text{if either } \frac{b_1-a_1}{\pi}\sqrt{\lambda} \le 1 \text{ or } \frac{b_2-a_2}{\pi}\sqrt{\lambda} \le 1 , \qquad N^{\mathbf{R}}(\lambda)=0 ; \qquad (2.6)$$

(ii) for $j\in\{2,3\}$, $N(\lambda,H^1_{S_j}(\mathbf{R}),L_2(\mathbf{R}),(\nabla\cdot,\nabla\cdot)_{\mathbf{R}}) = \#\{(k,l)\in\mathbb{N}\times\mathbb{N}_0 : \left(\frac{k\pi}{b_1-a_1}\right)^2+\left(\frac{(l+1/2)\pi}{b_2-a_2}\right)^2 \le\lambda\}$, the number of points $(k,l)\in\mathbb{N}\times\mathbb{N}_0$

inside or on the ellipse

$$\frac{k^2}{\lambda(b_1-a_1)^2/\pi^2}+\frac{(l+1/2)^2}{\lambda(b_2-a_2)^2/\pi^2}=1\ ,$$

from which follows

$$N(\lambda,H^1_{S_j}(R),L_2(R),(\nabla\cdot,\nabla\cdot)_R) \le \frac{\mathrm{Vol}(R)}{4\pi}\lambda + \frac{b_1-a_1}{2\pi}\sqrt{\lambda} \tag{2.7}$$

and,

$$\text{if either } \frac{b_1-a_1}{\pi}\sqrt{\lambda} \le 1 \text{ or } \frac{b_2-a_2}{\pi}\sqrt{\lambda} \le 1/2\,,\quad N(\lambda,H^1_{S_j}(R),L_2(R),(\nabla\cdot,\nabla\cdot)_R)=0\,. \tag{2.8}$$

3. Eigenvalues and disconnected domains

3.1. Given a non-void open bounded subset B of $\mathbb{R}^2$ we define for each $\varepsilon>0$ the *ε-neighbourhood of* ∂B *relative to* B by

$$\partial B_\varepsilon := \{x\in B : \mathrm{dist}(x,\partial B) < \varepsilon\}\,. \tag{3.1}$$

In the sequel there will be many statements of the form C(x) as $x\to l$, where C(x) is a condition involving the variable x. With this kind of statement we mean that C(x) is true at least if x is close enough to l.

For the meaning of $\approx$ and $\sim$ please refer to the Table of Notation.

3.2. Definition. (i) *For* $d\ge0$ *the* d-dimensional upper Minkowski content of ∂B relative to B *is given by*

$$M_d^B := \limsup_{\varepsilon\to0^+}\frac{\mathrm{Vol}(\partial B_\varepsilon)}{\varepsilon^{2-d}}\ ;$$

(ii) *the* Minkowski dimension of ∂B relative to B *is defined by*

$$D := D^B := \inf\{d\ge0 : M_d^B=0\} = \sup\{d\ge0 : M_d^B=\infty\}\ .$$

3.3. Definition. *Given a strictly decreasing function* a *defined on* $[n_0,\infty)$, *with* $n_0\in\mathbb{Z}$, *such that* $\lim_{x\to\infty} a(x) = 0$, *the* bounded open set associated with a *is*

$$B(a) := \bigcup_{i=n_0}^{\infty} R(a,i) := \bigcup_{i=n_0}^{\infty}]a(i+1),a(i)[\times]-1/2,1/2[\ .$$

This can be visualized by means of Figure 1, supposing that we continue to draw vertical segments as close to the x_2-axis as we wish.

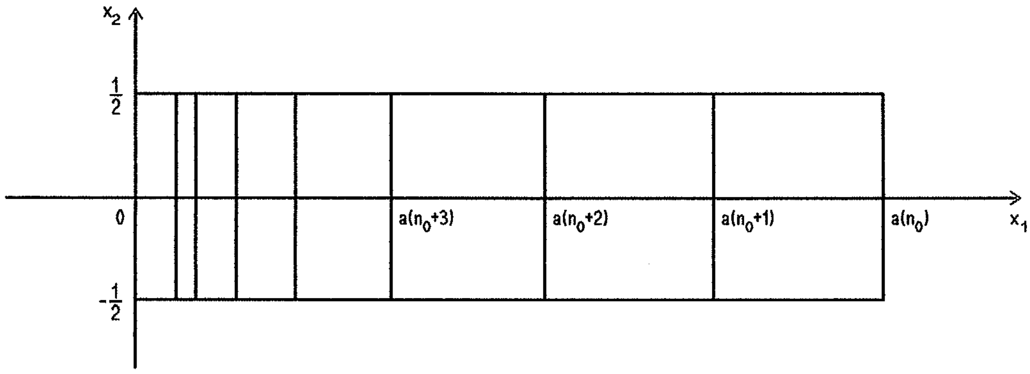

FIGURE 1

Since we would like to compare our special results with the general ones stated in Proposition 1.3, we need to compute both the relative Minkowski dimension and d-dimensional upper Minkowski contents of $\partial B(a)$. As a first step to achieve this we have

3.4. **Proposition.** *Let* a *be as in Definition 3.3 and suppose there exists a strictly decreasing function* $f:[n_0,\infty)\longrightarrow\mathbb{R}$ *such that* $f(x)\approx a(x)-a(x+1)$ as $x\to\infty$. *Then there exist constants* $c_1,c_2>0$ *such that for all small enough* $\varepsilon>0$,

$$\mathrm{Vol}(\partial B(a)_\varepsilon) \geq 2a(n_0)\varepsilon + a(f^{-1}(2c_1\varepsilon)+1).(1-2\varepsilon) + 2\varepsilon(1-2\varepsilon).(f^{-1}(2c_2\varepsilon)-n_0)$$

and

$$\mathrm{Vol}(\partial B(a)_\varepsilon) \leq 2a(n_0)\varepsilon + a(f^{-1}(2c_2\varepsilon)).(1-2\varepsilon) + 2\varepsilon(1-2\varepsilon).(f^{-1}(2c_1\varepsilon)-n_0+1)\,.$$

Proof. We consider $\varepsilon>0$ so small so that all of the following makes sense.

It is clear that

$$\partial B(a)_\varepsilon = \bigcup_{i=n_0}^{\infty} \partial R(a,i)_\varepsilon \tag{3.2a}$$

and

$$\mathrm{Vol}(\partial R(a,i)_\varepsilon) = \min\{a(i)-a(i+1),2(a(i)-a(i+1))\varepsilon+2\varepsilon-4\varepsilon^2\}\,. \tag{3.2b}$$

Defining $i_0(\varepsilon) = \min\{i\in\mathbb{Z}: i\geq n_0 \wedge a(i)-a(i+1)\leq 2\varepsilon\}-1$ we can write, in view of (3.2),

$$\begin{aligned}\mathrm{Vol}(\partial B(a)_\varepsilon) &= \sum_{i=n_0}^{i_0(\varepsilon)} \mathrm{Vol}(\partial R(a,i)_\varepsilon) + \sum_{i=i_0(\varepsilon)+1}^{\infty} \mathrm{Vol}(\partial R(a,i)_\varepsilon) \\ &= 2a(n_0)\varepsilon + a(i_0(\varepsilon)+1).(1-2\varepsilon) + 2\varepsilon(1-2\varepsilon).(i_0(\varepsilon)-n_0+1)\,.\end{aligned} \tag{3.3}$$

By the hypotheses on f there exist $c_1,c_2>0$ such that $c_1(a(x)-a(x+1)) \le f(x) \le c_2(a(x)-a(x+1))$ for large enough x, and it is not difficult to see that this implies that $f^{-1}(2c_2\varepsilon)-1 \le i_0(\varepsilon) \le f^{-1}(2c_1\varepsilon)$.

From this and (3.3) we obtain the desired result.

3.5. **PROPOSITION.** *Let* a *be as in Definition 3.3 and suppose there exists a strictly decreasing function* $f:[n_0,\infty)\longrightarrow \mathbb{R}$ *such that* $f(x)\approx a(x)-a(x+1)$ *as* $x\to\infty$. *Then there are constants* $c,c_1,c_2>0$ *such that*

$$\begin{aligned}\frac{\lambda}{4\pi}\,a(f^{-1}(c_1\pi/\sqrt{\lambda})+1) &\le \frac{\mathrm{Vol}(B(a))}{4\pi}\lambda - N^{B(a)}(\lambda) \\ &\le \frac{\lambda}{4\pi}\,a(f^{-1}(c_2\pi/\sqrt{\lambda})) + c.(f^{-1}(c_1\pi/\sqrt{\lambda}) - n_0 + 1)\sqrt{\lambda}\end{aligned}$$

for large enough λ.

Proof. We consider λ so large so that all of the following makes sense.

We use the following decomposition of $N^{B(a)}(\lambda)$ — recall Proposition I.4.18(ii):

$$N^{B(a)}(\lambda) = \sum_{i=n_0}^{i_1(\lambda)} N^{R(a,i)}(\lambda) + \sum_{i=i_1(\lambda)+1}^{i_2(\lambda)} N^{R(a,i)}(\lambda) + N^{T(a)}(\lambda)\,, \tag{3.4}$$

where $T(a) = \bigcup_{i=i_2(\lambda)+1}^{\infty} R(a,i)$,

$$i_1(\lambda) = \max\{i\in\mathbb{Z}: i\ge n_0 \wedge \frac{a(i)-a(i+1)}{\pi}\sqrt{\lambda}\ge 1\} \tag{3.5}$$

and $i_2(\lambda)$ is chosen so that $\mathrm{Vol}(T(a))<\pi/\sqrt{\lambda}$.

Using the definition of i_1 and (2.6) we immediately conclude that the second summation in (3.4) is zero. As to the third term on the right-hand side of (3.4), it does not exceed $N^R(\lambda)$, where here $R=]0,a(i_2(\lambda)+1)[\times]-1/2,1/2[$, — see Proposition I.4.17(ii) — and this in turn is zero, in view of the choice of i_2, again using (2.6). Hence

$$N^{B(a)}(\lambda) = \sum_{i=n_0}^{i_1(\lambda)} N^{R(a,i)}(\lambda) .$$

Using this and (2.5) we then obtain

$$\begin{aligned} \frac{\lambda}{4\pi}\left(\mathrm{Vol}(B(a)) - a(n_o) + a(i_1(\lambda)+1)\right) &\le \frac{\mathrm{Vol}(B(a))}{4\pi}\lambda - N^{B(a)}(\lambda) \\ &\le \frac{\lambda}{4\pi}\left(\mathrm{Vol}(B(a)) - a(n_o) + a(i_1(\lambda)+1)\right) + c.(i_1(\lambda) - n_o + 1)\sqrt{\lambda} , \end{aligned} \tag{3.6}$$

where $c>0$ is a constant independent of λ.

Now the hypotheses on f give, as in the proof of Proposition 3.4, that

$$f^{-1}(c_2\pi/\sqrt{\lambda}) - 1 \le i_1(\lambda) \le f^{-1}(c_1\pi/\sqrt{\lambda}) , \tag{3.7}$$

for some constants $c_1,c_2>0$, and this together with (3.6) yields the desired result.

3.6. DEFINITION. *A function* h *is said* to belong to class $\mathcal{H}$ *if* h *is positive, differentiable in an interval* $[n_0,\infty)$, *with* $n_0\in\mathbb{Z}$, *and* $\lim\limits_{x\to\infty} x\frac{h'(x)}{h(x)} = 0$.

3.7. LEMMA. *If* $h\in\mathcal{H}$ *then*

(i) *for any* $\varepsilon>0$ *there exists* $m(\varepsilon)>0$ *such that*

$$\left(\frac{z}{y}\right)^{-\varepsilon} \le \frac{h(z)}{h(y)} \le \left(\frac{z}{y}\right)^{\varepsilon} \quad \textit{whenever} \quad z\ge y\ge m(\varepsilon) ;$$

(ii) *for any* $\alpha>0$ $\quad h(x) = o(x^{\alpha})$ as $x\to\infty$;

(iii) *for any* $\alpha<0$ $\quad x^{\alpha} = o(h(x))$ as $x\to\infty$;

(iv) *for any bounded function* c $\quad h(x+c(x)) \sim h(x)$ as $x\to\infty$;

(v) *for any positive function* c *such that* $c(x)\approx 1$ as $x\to\infty$

$$h(c(x).x) \sim h(x) \quad \text{as} \quad x\to\infty .$$

Proof. (i) Since $\lim\limits_{x\to\infty} x\frac{h'(x)}{h(x)} = 0$ then

$$\forall\varepsilon>0,\ \exists m(\varepsilon)>0:\ x\ge m(\varepsilon) \Rightarrow -\varepsilon \le x\frac{h'(x)}{h(x)} \le \varepsilon .$$

Thus for $x\ge m(\varepsilon)$ we have $-\frac{\varepsilon}{x} \le \frac{h'(x)}{h(x)} \le \frac{\varepsilon}{x}$, and if $z\ge y\ge m(\varepsilon)$ we get

$$-\int_y^z \frac{\varepsilon}{x}\,dx \le \int_y^z \frac{h'(x)}{h(x)}\,dx \le \int_y^z \frac{\varepsilon}{x}\,dx,$$

from which follows $\left(\frac{z}{y}\right)^{-\varepsilon} \le \frac{h(z)}{h(y)} \le \left(\frac{z}{y}\right)^{\varepsilon}$.

(ii) to (v) follow from (i) by routine arguments.

3.8. LEMMA. *Let* a *be defined by* $a(x) = x^{\alpha}h(x)$, $x \in [n_0,\infty)$, $n_0 \in \mathbb{N}$, $\alpha<0$, *with* $h \in \mathcal{H}$. *For large enough* x *we have then that* a *is a strictly decreasing function such that* $\lim_{x\to\infty} a(x) = 0$. *Moreover,* $a(x)-a(x+1) \sim -\alpha x^{\alpha-1}h(x)$ as $x\to\infty$.

Proof. By Lemma 3.7(ii) we have $\frac{h(x)}{x^{-\alpha}} \longrightarrow 0$ as $x\to\infty$, from which follows that $\lim_{x\to\infty} a(x) = 0$.

Since $a'(x) = x^{\alpha-1}h(x)\left(\alpha + x\frac{h'(x)}{h(x)}\right)$ and $h \in \mathcal{H}$ then $a'(x)<0$ for all large enough x and therefore a(x) is strictly decreasing for large x.

Besides, $a(x)-a(x+1) = -\xi^{\alpha-1}h(\xi)\left(\alpha + \xi\frac{h'(\xi)}{h(\xi)}\right)$, with $\xi \in]x,x+1[$, by the mean value theorem, and, using Lemma 3.7(iv),

$$\frac{a(x)-a(x+1)}{-\alpha x^{\alpha-1}h(x)} = \frac{1}{\alpha}\left(\frac{\xi}{x}\right)^{\alpha-1}\frac{h(\xi)}{h(x)}\left(\alpha + \xi\frac{h'(\xi)}{h(\xi)}\right) \longrightarrow 1 \qquad \text{as} \quad x\to\infty.$$

3.9. Note that $-\alpha x^{\alpha-1}h(x) = x^{\alpha-1}(-\alpha h(x))$ is also of the form of the a of the preceding lemma, which means in particular that *it is also strictly decreasing* for large x. Actually, it is strictly decreasing wherever $x^{\alpha}h(x)$ is.

In the sequel, whenever we consider a of the form $a(x) = x^{\alpha}h(x)$, $x \in [n_0,\infty)$, $n_0 \in \mathbb{N}$, $\alpha<0$, with $h \in \mathcal{H}$, we suppose already that n_0 has been redefined so that a is strictly decreasing in $[n_0,\infty)$, and *we call* $\mathcal{A}$ *the class of all such functions.* Given $a \in \mathcal{A}$, f *will always denote the function defined by* $f(x) = -\alpha x^{\alpha-1}h(x)$, $x \in [n_0,\infty)$.

3.10. LEMMA. *Given* $a \in \mathcal{A}$ *and* $c>0$, $f^{-1}(cy) \approx f^{-1}(y)$ as $y\to 0^+$.

Proof. That f is invertible is a consequence of a being in $\mathcal{A}$ and of 3.9. Actually, f is strictly decreasing and $\lim_{x\to\infty} f(x) = 0$.

Given a positive constant c, let $b_1 = (2c)^{1/(\alpha-1)}$ and $b_2 = \left(\frac{c}{2}\right)^{1/(\alpha-1)}$. Then, using Lemma 3.7(v),

$$\lim_{x\to\infty} \frac{f(b_1x)}{f(x)} = b_1^{\alpha-1} > c \quad \text{and} \quad \lim_{x\to\infty} \frac{f(b_2x)}{f(x)} = b_2^{\alpha-1} < c,$$

so that $f(b_1x) > c.f(x) > f(b_2x)$ for $x>x_0$ say. Now, if $0<y<f(x_0)$ then $f^{-1}(y) > x_0$ and therefore $f(b_1.f^{-1}(y)) > cy > f(b_2.f^{-1}(y))$, that is, $b_1.f^{-1}(y) < f^{-1}(cy) < b_2.f^{-1}(y)$.

3.11. **Proposition.** *Given* $a\in\mathcal{A}$ *we have*

$$\frac{\mathrm{Vol}(B(a))}{4\pi}\lambda - N^{B(a)}(\lambda) \approx \lambda^{1/2} f^{-1}(\lambda^{-1/2}) \quad \text{as} \quad \lambda\to\infty .$$

Proof. Note first that we can use Proposition 3.5, for all the hypotheses considered there are satisfied here.

Using the definition of f and Lemmas 3.7(iv) and 3.10 we can write

$$\begin{aligned} a(f^{-1}(c_1\pi/\sqrt{\lambda})+1) &= (-\alpha)^{-1}(f^{-1}(c_1\pi/\sqrt{\lambda})+1).f(f^{-1}(c_1\pi/\sqrt{\lambda})+1) \\ &\geq c_3 f^{-1}(\lambda^{-1/2})\lambda^{-1/2} \quad \text{as} \quad \lambda\to\infty, \end{aligned} \tag{3.8}$$

where c_1 and c_3 are positive constants. Analogously,

$$a(f^{-1}(c_2\pi/\sqrt{\lambda})) \leq c_4 f^{-1}(\lambda^{-1/2})\lambda^{-1/2} \quad \text{as} \quad \lambda\to\infty,$$

with c_2 and c_4 positive constants.

Using this information in the conclusion of Proposition 3.5 together with Lemma 3.10, we can say that there are $c_5,c_6,c_7>0$ such that

$$\begin{aligned} c_5\lambda^{1/2}f^{-1}(\lambda^{-1/2}) &\leq \frac{\mathrm{Vol}(B(a))}{4\pi}\lambda - N^{B(a)}(\lambda) \\ &\leq c_6\lambda^{1/2}f^{-1}(\lambda^{-1/2}) + c_7\lambda^{1/2}f^{-1}(\lambda^{-1/2}) \quad \text{as} \quad \lambda\to\infty, \end{aligned}$$

and our proof is complete.

The proposition just proved shows that any possible second term in the asymptotic expansion for $N^{B(a)}(\lambda)$ depends heavily on the way in which the width of the rectangles R(a,i) approaches zero as i goes to infinity. Before proceeding to more concrete examples of functions a satisfying Proposition

3.11, let us derive from Proposition 3.4 handy estimates to compute the relative Minkowski dimension of $\partial B(a)$.

3.12. **Proposition.** *Given* $a \in \mathcal{A}$ *and* $d \in \mathbb{R}$ *we have*

$$\frac{\mathrm{Vol}(\partial B(a)_\varepsilon)}{\varepsilon^{2-d}} \approx \varepsilon^{d-1} f^{-1}(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0^+ .$$

Proof. The hypotheses of Proposition 3.4 are obviously satisfied. Using that proposition and Lemma 3.10, the proof is completely analogous to the preceding one until the point at which we see that there exist $c_1, c_2, c_3, c_4 > 0$ such that

$$c_1 \varepsilon f^{-1}(\varepsilon) + c_2 \varepsilon f^{-1}(\varepsilon) \le \mathrm{Vol}(\partial B(a)_\varepsilon) \le c_3 \varepsilon f^{-1}(\varepsilon) + c_4 \varepsilon f^{-1}(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0^+ ,$$

and from here we obtain the desired result by dividing all members of these inequalities by ε^{2-d}.

Next we give an useful criterion for the estimation of f^{-1}.

3.13. **Proposition.** *Let* $a \in \mathcal{A}$ *and assume that*

$$h(\lambda^{1/(2-2\alpha)} h(\lambda)^{1/(1-\alpha)}).h(\lambda)^{-1} \approx 1 \qquad \text{as} \quad \lambda \to \infty .$$

Then

$$f^{-1}(\lambda^{-1/2}) \approx \lambda^{1/(2-2\alpha)} h(\lambda)^{1/(1-\alpha)} \qquad \text{as} \quad \lambda \to \infty .$$

Proof. By hypothesis there are $c_1, c_2 > 0$ such that

$$c_1 \le h(\lambda^{1/(2-2\alpha)} h(\lambda)^{1/(1-\alpha)}).h(\lambda)^{-1} \le c_2 \qquad \text{for large } \lambda ,$$

from which follows

$$-\alpha\, c_1 \lambda^{-1/2} \le f(\lambda^{1/(2-2\alpha)} h(\lambda)^{1/(1-\alpha)}) \le -\alpha\, c_2 \lambda^{-1/2}$$

and, with the help of Lemma 3.10,

$$c_3 f^{-1}(\lambda^{-1/2}) \ge \lambda^{1/(2-2\alpha)} h(\lambda)^{1/(1-\alpha)} \ge c_4 f^{-1}(\lambda^{-1/2}) ,$$

for some positive constants c_3 and c_4.

3.14. **Corollary.** *Let* $a\in\mathcal{A}$ *and assume that*

$$h(\lambda^{1/(2-2\alpha)}h(\lambda)^{1/(1-\alpha)}).h(\lambda)^{-1} \approx 1 \qquad \text{as} \quad \lambda\to\infty .$$

Then

$$\frac{\mathrm{Vol}(B(a))}{4\pi}\lambda - N^{B(a)}(\lambda) \approx \lambda^{(2-\alpha)/(2-2\alpha)}h(\lambda)^{1/(1-\alpha)} \qquad \text{as} \quad \lambda\to\infty .$$

Proof. This is a straightforward consequence of Propositions 3.11 and 3.13.

This result is decisive in showing that any possible second term in the asymptotic expansion for $N^{B(a)}(\lambda)$ need not be of power form. For a family of explicit examples where this is really the case, consider $h(x) = \log x$. We shall return later on to establish this also for other functions h, but first let us compute $D^{B(a)}$ for a as in the corollary above.

3.15. **Proposition.** *Let* $a\in\mathcal{A}$ *be as in Corollary 3.14. Then the Minkowski dimension of* $\partial B(a)$ *relative to* $B(a)$ *is* $\frac{2-\alpha}{1-\alpha}$, *and* $M^{B(a)}_{(2-\alpha)/(1-\alpha)}$ *can differ from* $\limsup_{\varepsilon\to0^+} h(\varepsilon^{-2})^{1/(1-\alpha)}$ *only by a positive (and finite) multiplicative factor.*

Proof. Proposition 3.13 enables us to write

$$f^{-1}(\varepsilon) \approx \varepsilon^{1/(\alpha-1)}h(\varepsilon^{-2})^{1/(1-\alpha)} \qquad \text{as} \quad \varepsilon\to0^+ .$$

Using this in the conclusion of Proposition 3.12 we have

$$\frac{\mathrm{Vol}(\partial B(a)_\varepsilon)}{\varepsilon^{2-d}} \approx \varepsilon^{d-1+1/(\alpha-1)}h(\varepsilon^{-2})^{1/(1-\alpha)} \qquad \text{as} \quad \varepsilon\to0^+ \tag{3.9}$$

for $d\in\mathbb{R}$. Writing $d = \frac{2-\alpha}{1-\alpha} + \eta$, with $\eta\in\mathbb{R}$, we obtain then

$$\frac{\mathrm{Vol}(\partial B(a)_\varepsilon)}{\varepsilon^{2-d}} \approx \varepsilon^{\eta}h(\varepsilon^{-2})^{1/(1-\alpha)} = ((\varepsilon^{-2})^{-\eta(1-\alpha)/2}h(\varepsilon^{-2}))^{1/(1-\alpha)} \qquad \text{as} \quad \varepsilon\to0^+ ,$$

and this approaches $\begin{cases} 0 & \text{if } \eta>0 \\ \infty & \text{if } \eta<0 \end{cases}$ as $\varepsilon\to0^+$, taking account of Lemma 3.7(ii),(iii).

Thus, by definition, $D^{B(a)} = \frac{2-\alpha}{1-\alpha}$.

Substituting $d = \frac{2-\alpha}{1-\alpha}$ in (3.9) and applying $\limsup_{\varepsilon \to 0^+}$ we have the second part of the proposition.

In the sequel we shall use the notation $\log^N x$, for $N \in \mathbb{N}_0$, with the meaning of composition of functions, that is, $\log^N x = \log(\log^{N-1} x)$ if $N \neq 0$ and $\log^0 x = x$.

3.16. **Proposition.** *Given* $\beta \in \mathbb{R}$ *and* $N \in \mathbb{N}$, *the function* h *given by* $h(x) = (\log^N x)^\beta$ *in any interval* $[n_0, \infty)$, *with* $n_0 \in \mathbb{N}$, *where it is defined and positive, belongs to class* $\mathcal{H}$ *and, for any* $\alpha<0$, $h(\lambda^{1/(2-2\alpha)} h(\lambda)^{1/(1-\alpha)}).h(\lambda)^{-1}$ *approaches* $c^\beta_{\alpha,N}$ *as* $\lambda \to \infty$, *where*

$$c^\beta_{\alpha,N} = \begin{cases} (2(1-\alpha))^{-\beta} & \text{if } N=1 \\ 1 & \text{if } N>1 \end{cases} .$$

Proof. Since the case $\beta=0$ is trivial, we consider $\beta \neq 0$.

First of all, there are plenty of intervals $[n_0, \infty)$ where h is defined and positive. In such an interval it is also differentiable, of course.

To complete the proof that $h \in \mathcal{H}$, use induction on N.

As to the remaining part of the proposition, it is clear that it is enough to prove the stronger result

$$\frac{\log^N(\lambda^{1/(2-2\alpha)}(\log^M \lambda)^{\beta/(1-\alpha)})}{\log^N \lambda} \longrightarrow c_{\alpha,N} \quad \text{as} \quad \lambda \to \infty , \tag{3.10}$$

for all $\alpha<0$, $M \in \mathbb{N}$ and all β and N. And this can be done in a straightforward manner by using L'Hospital's rule and again induction on N and for all α, β and M at any stage. We then obtain firstly that (3.10) is true for N=1 not only as it is stated above but also when we substitute the two functions in the fraction by the corresponding derivatives; and, secondly, granted that both the fraction in (3.10) and the fraction with the corresponding derivatives approach the same constant c at a stage N=k, then they will both approach 1 at the stage N=k+1.

This concludes the proof.

3.17. COROLLARY. *Let* $a \in \mathcal{A}$ *be given by* $a(x) = x^{\alpha}(\log^N x)^{\beta}$, *for* α, β *and* N *fixed respectively in* $\mathbb{R}^-$, $\mathbb{R}$ *and* $\mathbb{N}$. *Then*

$$\frac{\mathrm{Vol}(B(a))}{4\pi}\lambda - N^{B(a)}(\lambda) \approx \lambda^{(2-\alpha)/(2-2\alpha)}(\log^N\lambda)^{\beta/(1-\alpha)} \qquad \text{as} \quad \lambda\to\infty. \tag{3.11}$$

Moreover, $D^{B(a)} = \frac{2-\alpha}{1-\alpha}$ *and*

$$M^{B(a)}_{(2-\alpha)/(1-\alpha)} = \begin{cases} \infty & \text{if } \beta>0 \\ \text{finite value } \neq 0 & \text{if } \beta=0\ . \\ 0 & \text{if } \beta<0 \end{cases} \tag{3.12}$$

Proof. (3.11) is a direct consequence of the definition of $\mathcal{A}$ — see 3.9 —, Proposition 3.16 and Corollary 3.14. The result about the relative Minkowski dimension is also immediate from Proposition 3.15, and this also implies (3.12).

3.18. Some comments are in order, with respect to the results just proved.

If $\beta=0$ we have the case pointed out in [Lapidus a, Ex. 5.1']. If $\beta\neq 0$ we get examples where the results of Lapidus [a, Th. 2.3] — see our Proposition 1.3 for the statement in our context — are not sharp. Further, if we consider only $\beta>0$ then $M^{B(a)}_{(2-\alpha)/(1-\alpha)} = \infty$ and we have what Lapidus called *degenerate* cases and for which he only says that for $d > \frac{2-\alpha}{1-\alpha}$

$$\frac{\mathrm{Vol}(B(a))}{4\pi}\lambda - N^{B(a)}(\lambda) = O(\lambda^{d/2}) \qquad \text{as} \quad \lambda\to\infty \tag{3.13}$$

— in fact, in these cases it is really possible for (3.13) not to be true when $d = \frac{2-\alpha}{1-\alpha}$, as is shown by (3.11) when $\beta>0$.

Corollary 3.17 also shows that it is not an accident that the second term in the asymptotic expansion for $N^{B(a)}$ is not of power form when $\beta\neq 0$. And the forms it can assume seem to depend heavily on the details of the boundary of the domain in question — this will be more clear after we present the examples with connected sets in the next section.

It seems impossible to include the limit case $\alpha=0$ in the preceding results without losing some harmony in the proofs, since a function a in this case need not have the nice properties of functions belonging to class $\mathcal{A}$. However, by mimicking some steps carried out in the arguments given, it is possible to prove some interesting results in this situation too. As an example, we prove the following

3.19. **Proposition.** *Let* a *be given by* $a(x)=\frac{1}{\log x}$ *on* $[2,\infty)$. *Then the conclusion of Corollary 3.17 still holds if we substitute* α, β *and* N *respectively by* 0, -1 *and* 1, *that is,*

$$\frac{\mathrm{Vol}(B(a))}{4\pi}\lambda - N^{B(a)}(\lambda) \approx \frac{\lambda}{\log\lambda} \qquad \text{as} \quad \lambda\to\infty,$$

$D^{B(a)}=2$ *and* $M_2^{B(a)}=0$.

Proof. Note that a is strictly decreasing and $\lim_{x\to\infty} a(x) = 0$, so that we can consider the bounded open set associated with a in the sense of Definition 3.3. Note also that a belongs to class $\mathcal{H}$.

Using the mean value theorem we have, with $\xi\in]x,x+1[$,

$$\frac{a(x)-a(x+1)}{x^{-1}(\log x)^{-2}} = \left(\frac{x}{\xi}\right)\left(\frac{\log x}{\log \xi}\right)^2 \longrightarrow 1 \qquad \text{as} \quad x\to\infty,$$

that is,

$$a(x)-a(x+1) \sim \frac{1}{x(\log x)^2} \qquad \text{as} \quad x\to\infty.$$

Since $f(x):=\frac{1}{x(\log x)^2}$ is strictly decreasing, it is invertible; moreover, given any $c>0$ we have $f^{-1}(cy)\approx f^{-1}(y)$ as $y\to 0^+$, the proof of this being analogous to the proof of Lemma 3.10 with $\alpha=0$.

We are now in a position to use Propositions 3.4 and 3.5 and, reasoning as in the proofs of Propositions 3.11 and 3.12, obtain

$$\frac{\mathrm{Vol}(B(a))}{4\pi}\lambda - N^{B(a)}(\lambda) \approx \lambda^{1/2} f^{-1}(\lambda^{-1/2}) \log f^{-1}(\lambda^{-1/2}) \qquad \text{as} \quad \lambda\to\infty \tag{3.14}$$

and

$$\frac{\mathrm{Vol}(\partial B(a)_\varepsilon)}{\varepsilon^{2-d}} \approx \varepsilon^{d-1} f^{-1}(\varepsilon) \log f^{-1}(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0^+ . \tag{3.15}$$

Note now that, as $\lambda\to\infty$,

$$\left(\log \frac{\lambda^{1/2}}{(\log\lambda)^2}\right)^{-1} . \log\lambda = \frac{\log\lambda}{(\log\lambda)/2 - 2\log(\log\lambda)} \longrightarrow 2 , \tag{3.16}$$

therefore

$$3\,\lambda^{-1/2} \le f\Big(\frac{\lambda^{1/2}}{(\log\lambda)^2}\Big) \le 5\,\lambda^{-1/2} ,$$

that is,

$$f^{-1}(\lambda^{-1/2}) \approx \frac{\lambda^{1/2}}{(\log\lambda)^2} . \tag{3.17}$$

This in turn implies, in virtue of (3.16), that

$$\log f^{-1}(\lambda^{-1/2}) \approx \log\lambda \qquad \text{as} \quad \lambda\to\infty . \tag{3.18}$$

Putting (3.17) and (3.18) in (3.14) and (3.15) gives

$$\frac{\mathrm{Vol}(B(a))}{4\pi}\lambda - N^{B(a)}(\lambda) \approx \frac{\lambda}{\log\lambda} \qquad \text{as} \quad \lambda\to\infty$$

and

$$\frac{\mathrm{Vol}(\partial B(a)_\varepsilon)}{\varepsilon^{2-d}} \approx \frac{\varepsilon^{d-2}}{\log(\varepsilon^{-2})} \qquad \text{as} \quad \varepsilon\to 0^+ ,$$

which proves part of the proposition. Reasoning about the latter formula as in the proof of Proposition 3.15, with $\alpha=0$ and $h(x) = \frac{1}{\log x}$, proves the remaining part.

Note that Proposition 3.19 gives an example of what can happen in the very *fractal* case, for the Minkowski dimension of $\partial B(a)$ relative to the B(a) just considered equals 2.

4. Eigenvalues and connected domains

4.1. Suppose that instead of the set B(a) of Definition 3.3 we consider B(a,l) defined by

$$B(a,l) := \,]0,a(n_0)[\times]-1/2,1/2[\, \setminus \bigcup_{i=n_0+1}^{\infty} \{a(i)\}\times[l,1/2) \;, \tag{4.1}$$

where l is a fixed number belonging to $]-1/2,1/2[$.

This can be visualized by means of Figure 2, supposing that we continue to draw vertical segments as close to the x_2-axis as we wish.

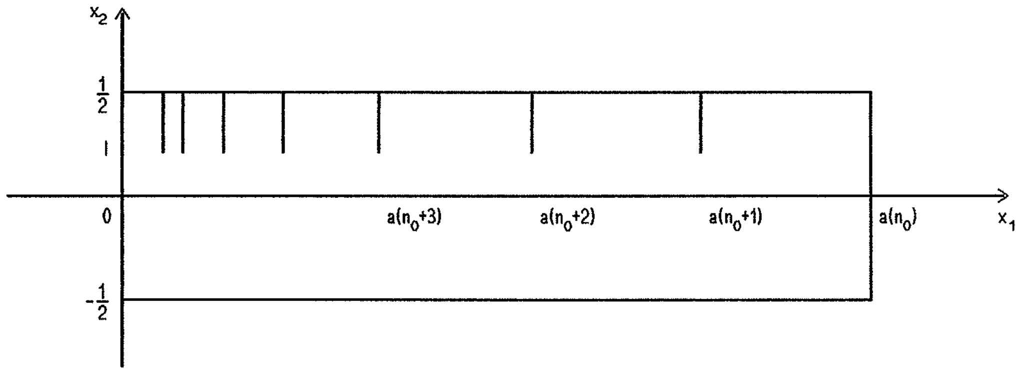

FIGURE 2

$B(a,l)$ is an open bounded connected subset of $\mathbb{R}^2$ which satisfies

$$R(a) := \,]0,a(n_0)[\times]-1/2,1/2[\, \supset B(a,l) \supset B(a)$$

and consequently

$$N^{R(a)}(\lambda) \geq N^{B(a,l)}(\lambda) \geq N^{B(a)}(\lambda) \tag{4.2}$$

— see Proposition I.4.17(ii). If $a\in\mathcal{A}$ we can then use Proposition 3.11 and (2.5) to get, with a constant $c>0$,

$$0 \leq \frac{\mathrm{Vol}(B(a,l))}{4\pi}\lambda - N^{B(a,l)}(\lambda) \leq c\,\lambda^{1/2} f^{-1}(\lambda^{-1/2}) \qquad \text{as} \quad \lambda\to\infty\,. \tag{4.3}$$

On the other hand we can prove that

$$\begin{aligned} a(n_0)\varepsilon + a(f^{-1}(2c_1\varepsilon)+1).(1/2-l-\varepsilon) + 2\varepsilon(1/2-l-\varepsilon).(f^{-1}(2c_2\varepsilon)-n_0) \leq \\ \leq \mathrm{Vol}(\partial B(a,l)_\varepsilon) \leq \mathrm{Vol}(\partial B(a)_\varepsilon) \end{aligned} \tag{4.4}$$

for $\varepsilon>0$ small enough, where the symbols on the left-hand side are exactly the same as in Proposition 3.4, and where the first inequality above can be obtained in a way similar to that used to prove the corresponding inequality in that proposition. Moreover, if $a \in \mathcal{A}$ we can even write

$$c.\mathrm{Vol}(\partial B(a)_\varepsilon) \le \mathrm{Vol}(\partial B(a,l)_\varepsilon) \le \mathrm{Vol}(\partial B(a)_\varepsilon) \qquad \text{as} \quad \varepsilon \to 0^+, \tag{4.5}$$

for some positive constant c, so that if we assume the hypotheses of Proposition 3.15 to hold, we then have

$$D^{B(a,l)} = \frac{2-\alpha}{1-\alpha}, \text{ and } M^{B(a,l)}_{(2-\alpha)/(1-\alpha)} \text{ can differ from } \limsup_{\varepsilon\to 0^+} h(\varepsilon^{-2})^{1/(1-\alpha)} \text{ only by a positive (and finite) multiplicative factor.} \tag{4.6}$$

4.2. We want in this subsection to obtain a result for $N^{B(a,l)}$ such as Proposition 3.11 for $N^{B(a)}$. In view of (4.3) above, we only need to obtain a corresponding lower estimate.

We fix a and l as in the beginning of 4.1, assume that λ is large enough, and introduce the following notation (valid only for the present subsection):

$i_1(\lambda)$ has the same meaning as in (3.5);

$B := B(a,l)$;

$B' := \{(x_1,x_2) \in B(a,l) : x_2 > l\}$;

$R_- := \{(x_1,x_2) \in B(a,l) : x_2 < l\} =]0,a(n_0)[\times]-1/2,l[$;

$R_i :=]a(i+1),a(i)[\times]l,1/2[$, $i = n_0, \dots, \infty$;

$T(\lambda) := \bigcup_{i=i_2(\lambda)+1}^{\infty} R_i$, where $i_2(\lambda)$ is chosen so that $\mathrm{Vol}(T(\lambda)) < \frac{\pi}{\sqrt{\lambda}}(1/2-l)$;

$R(\lambda) :=]0,a(i_2(\lambda)+1)[\times]l,1/2[$;

$S := \partial B$;

$S' := \partial B' \setminus \bigcup_{i=n_0}^{\infty}]a(i+1),a(i)[\times\{l\}$;

$S_- := \partial R_- \setminus]0,a(n_0)[\times\{l\}$;

$S_i := \partial R_i \setminus]a(i+1),a(i)[\times\{l\}$, $i = n_0, \dots, \infty$;

$S_T := \partial T(\lambda) \setminus \bigcup_{i=i_2(\lambda)+1}^{\infty}]a(i+1),a(i)[\times\{l\}$;

$S_R := \partial R(\lambda) \setminus]0,a(i_2(\lambda)+1)[\times\{l\}$.

We can then write, using Propositions I.4.17(i) and I.4.18(i),

$$N^B(\lambda) = N(\lambda, H^1_S(B), L_2(B), (\nabla\cdot,\nabla\cdot)_B)$$
$$\le N(\lambda, H^1_{S_-}(R_-), L_2(R_-), (\nabla\cdot,\nabla\cdot)_{R_-}) + N(\lambda, H^1_{S'}(B'), L_2(B'), (\nabla\cdot,\nabla\cdot)_{B'}) \tag{4.7}$$

and

$$N(\lambda, H^1_{S'}(B'), L_2(B'), (\nabla\cdot,\nabla\cdot)_{B'}) \le \sum_{i=n_0}^{i_1(\lambda)} N(\lambda, H^1_{S_i}(R_i), L_2(R_i), (\nabla\cdot,\nabla\cdot)_{R_i}) +$$
$$+ \sum_{i=i_1(\lambda)+1}^{i_2(\lambda)} N(\lambda, H^1_{S_i}(R_i), L_2(R_i), (\nabla\cdot,\nabla\cdot)_{R_i}) + N(\lambda, H^1_{S_T}(T(\lambda)), L_2(T(\lambda)), (\nabla\cdot,\nabla\cdot)_{T(\lambda)}), \tag{4.8}$$

where the second summation is zero if $i_2(\lambda) = i_1(\lambda)$.

Now, by Proposition I.4.17(ii),

$$N(\lambda, H^1_{S_T}(T(\lambda)), L_2(T(\lambda)), (\nabla\cdot,\nabla\cdot)_{T(\lambda)}) \le N(\lambda, H^1_{S_R}(R(\lambda)), L_2(R(\lambda)), (\nabla\cdot,\nabla\cdot)_{R(\lambda)});$$

if we put this together with (4.8) and (4.7) and use the estimates (2.7) and (2.8), we get

$$N^B(\lambda) \le \frac{a(n_0).(l+1/2)}{4\pi}\lambda + \frac{a(n_0)}{2\pi}\sqrt{\lambda} + \sum_{i=n_0}^{i_1(\lambda)}\Big(\frac{(a(i)-a(i+1)).(1/2-l)}{4\pi}\lambda + \frac{a(i)-a(i+1)}{2\pi}\sqrt{\lambda}\Big)$$
$$= \frac{a(n_0).(l+1/2)}{4\pi}\lambda + \frac{a(n_0)}{2\pi}\sqrt{\lambda} + \frac{1/2-l}{4\pi}(a(n_0)-a(i_1(\lambda)+1))\lambda + \frac{1}{2\pi}(a(n_0)-a(i_1(\lambda)+1))\sqrt{\lambda}$$

and

$$\frac{\mathrm{Vol}(B)}{4\pi}\lambda - N^B(\lambda) \ge -\frac{a(n_0)}{\pi}\sqrt{\lambda} + \frac{1/2-l}{4\pi}a(i_1(\lambda)+1)\lambda + \frac{1}{2\pi}a(i_1(\lambda)+1)\sqrt{\lambda}$$
$$\ge \frac{1/2-l}{4\pi}a(i_1(\lambda)+1)\lambda - \frac{a(n_0)}{\pi}\sqrt{\lambda}\ . \tag{4.9}$$

Assuming now that $a\in\mathcal{A}$, we can estimate i_1 in terms of f^{-1} following (3.7), that is, $i_1(\lambda) \le f^{-1}(c_1\pi/\sqrt{\lambda})$, and, in turn, use the estimate $a(f^{-1}(c_1\pi/\sqrt{\lambda})+1) \ge c_3 f^{-1}(\lambda^{-1/2})\lambda^{-1/2}$ as set out in (3.8), so that (4.9) yields

$$\frac{\mathrm{Vol}(B)}{4\pi}\lambda - N^B(\lambda) \ge \Big(\frac{c_3}{4\pi}(1/2-l) - \frac{a(n_0)}{\pi f^{-1}(\lambda^{-1/2})}\Big).f^{-1}(\lambda^{-1/2})\lambda^{1/2}$$
$$\ge c_4\lambda^{1/2}f^{-1}(\lambda^{-1/2}) \qquad \text{as } \lambda\to\infty$$

for some constant $c_4>0$.

This together with (4.3) proves that Proposition 3.11 also holds for the sets B(a,l) considered here; that is, if we define, naturally, B(a,-1/2) = B(a), we have

4.3. **Proposition.** *Given* $a\in\mathcal{A}$ *and* $l\in[-1/2,1/2)$, *the following holds:*

$$\frac{\mathrm{Vol}(B(a,l))}{4\pi}\lambda - N^{B(a,l)}(\lambda) \approx \lambda^{1/2} f^{-1}(\lambda^{-1/2}) \qquad \text{as} \quad \lambda\to\infty .$$

4.4. As in section 3, we can specialize this result, so that Corollaries 3.14 and 3.17 (the latter with the help of (4.6)) hold with B(a) replaced by B(a,l), $l\in[-1/2,1/2)$.

Also the same is true of Proposition 3.19. In fact, the only things to be modified in the proof of this proposition are the way (3.14) and (3.15) are derived, because the rest does not take into account whether we are considering $l=-1/2$ or not; to show that (3.14) is true in our present context, use (4.2) to reduce the proof of the upper estimate to the case $l=-1/2$, and (4.9) and (3.7) for the lower bound; as to (3.15), use (4.4).

It is now clear that the *non-standard* behaviour of the remainder in the asymptotic expansion for $N^{B(a)}$ in the preceding section has nothing to do with the lack of connectedness of B(a), for we have the same behaviour for $N^{B(a,l)}$ when $l\in]-1/2,1/2[$, and in this case the B(a,l) are connected. The important characteristic seems to be what is going on on the boundary.

We shall now consider the partition function defined in (1.4). The results will follow immediately from the preceding ones through the following Abelian argument:

4.5. **Proposition.** *Given a non-void open bounded subset* B *of* $\mathbb{R}^2$, $\gamma>0$, $L=\lim_{\lambda\to\infty}\frac{N^B(\lambda)}{\lambda}$ *and* $h\in\mathcal{H}$, *if there exists* $c>0$ *such that*

$$L\lambda - N^B(\lambda) \le c\,\lambda^{\gamma}h(\lambda) \qquad (\text{respectively} \ \ge c\,\lambda^{\gamma}h(\lambda)) \qquad \text{as} \quad \lambda\to\infty \tag{4.10}$$

then there also exists $c'>0$ *such that*

$$Lt^{-1} - Z^B(t) \le c' t^{-\gamma}h(t^{-1}) \qquad (\text{respectively} \ \ge c' t^{-\gamma}h(t^{-1})) \qquad \text{as} \quad t\to 0^+ . \tag{4.11}$$

Proof. We consider only the case of inequality $\leq$, since the other can be dealt with similarly.

In the sequel all integrals are finite, although we shall not make any further mention of that.

Using Lebesgue-Stieltjes integration by parts, we can write

$$Z(t) = t\int_0^\infty e^{-\lambda t} N(\lambda)\, d\lambda \ .$$

From the hypotheses it follows that there is $\lambda_1 > m(1)$ — where $m(1)$ is $m(\varepsilon)$ for $\varepsilon=1$ as in Lemma 3.7(i) — such that the inequality in (4.10) holds for $\lambda \geq \lambda_1$. We can thus write, with a change of variables,

$$\begin{aligned} Z(t) &\geq t\int_{\lambda_1}^\infty e^{-\lambda t} N(\lambda)\, d\lambda \\ &\geq L\int_0^\infty e^{-\mu}\mu\, d\mu . t^{-1} - L\int_0^{\lambda_1 t} e^{-\mu}\mu\, d\mu . t^{-1} - c\int_{\lambda_1 t}^\infty e^{-\mu}\mu^\gamma h(\mu t^{-1})\, d\mu . t^{-\gamma} \ . \end{aligned} \tag{4.12}$$

The second term on the right-hand side is not less than

$$-c_1 t^{-\gamma} h(t^{-1}) \qquad \text{for } t<t_0 \text{ say (with } c_1 \text{ a positive constant)}, \tag{4.13}$$

taking account of Lemma 3.7(iii).

Now we assume that $t<\lambda_1^{-1}$ and write the last term in (4.12) as

$$-c\int_{\lambda_1 t}^1 e^{-\mu}\mu^\gamma h(\mu t^{-1})\, d\mu . t^{-\gamma} - c\int_1^\infty e^{-\mu}\mu^\gamma h(\mu t^{-1})\, d\mu . t^{-\gamma} . \tag{4.14}$$

Using Lemma 3.7.(i) with $\varepsilon=1$, $y=\mu t^{-1}$, $z=t^{-1}$, we can say, due to our choice of λ_1, that the first term in (4.14) is not less than

$$-c\int_{\lambda_1 t}^1 e^{-\mu}\mu^{\gamma-1}\, d\mu . t^{-\gamma} h(t^{-1}) \geq -c\int_0^1 e^{-\mu}\mu^{\gamma-1}\, d\mu . t^{-\gamma} h(t^{-1}) \ . \tag{4.15}$$

As to the second term in (4.14), we can prove in a similar way (now with $\varepsilon=1$, $y=t^{-1}$, $z=\mu t^{-1}$ in Lemma 3.7(i)) that it is not less than

$$-c\int_1^\infty e^{-\mu}\mu^{\gamma+1}\, d\mu . t^{-\gamma} h(t^{-1}) \ .$$

This and (4.12) to (4.15) show that for $t < \lambda_1^{-1}, t_o$

$$Z(t) \geq Lt^{-1} - c_1 t^{-\gamma} h(t^{-1}) - c_2 t^{-\gamma} h(t^{-1}) - c_3 t^{-\gamma} h(t^{-1}),$$

with constants $c_2, c_3 > 0$, which proves (4.11).

As mentioned above, this proposition enable us to translate for the partition function results already obtained for the counting function. Thus, in view of what is stated in 4.4, we have the following counterparts of Corollaries 3.14, 3.17 and Proposition 3.19, the proofs being trivial now:

4.6. COROLLARY. *Let* $a \in \mathcal{A}$, $l \in [-1/2, 1/2)$ *and*

$$h(\lambda^{1/(2-2\alpha)} h(\lambda)^{1/(1-\alpha)}).h(\lambda)^{-1} \approx 1 \qquad \text{as} \quad \lambda \to \infty .$$

Then

$$\frac{\mathrm{Vol}(B(a,l))}{4\pi} t^{-1} - Z^{B(a,l)}(t) \approx t^{-(2-\alpha)/(2-2\alpha)} h(t^{-1})^{1/(1-\alpha)} \qquad \text{as} \quad t \to 0^+ .$$

4.7. COROLLARY. *Let* $a \in \mathcal{A}$ *be given by* $a(x) = x^{\alpha}(\log^N x)^{\beta}$, *for* α, β *and* N *fixed respectively in* $\mathbb{R}^-$, $\mathbb{R}$ *and* $\mathbb{N}$. *Let* $l \in [-1/2, 1/2)$. *Then*

$$\frac{\mathrm{Vol}(B(a,l))}{4\pi} t^{-1} - Z^{B(a,l)}(t) \approx t^{-(2-\alpha)/(2-2\alpha)} (\log^N(t^{-1}))^{\beta/(1-\alpha)} \qquad \text{as} \quad t \to 0^+ .$$

4.8. COROLLARY. *Let* a *be given by* $a(x) = \frac{1}{\log x}$ *on* $[2, \infty)$, *and* $l \in [-1/2, 1/2)$.

Then

$$\frac{\mathrm{Vol}(B(a,l))}{4\pi} t^{-1} - Z^{B(a,l)}(t) \approx \frac{1}{t.\log(t^{-1})} \qquad \text{as} \quad t \to 0^+ .$$

CHAPTER IV

EIGENVALUES OF OPERATORS MODELLED ON THE STOKES OPERATOR

1. Introduction

1.1. In this chapter we give a small contribution to the problem of finding estimates for the remainder in the asymptotic formula for the counting functions associated with a family of operators which includes the Stokes operator (recall III.1.4 for the meaning of the expression just used, with obvious modifications due to the new context, of course).

Our approach starts off from what was done in [Métivier 1978], so we shall recall the setting used there. To avoid repetition, we shall refer to section 2 below, where the periodic case is treated.

Let Ω be a non-void open bounded Lipschitz subset of $\mathbb{R}^n$. Let N, J, m, m_j, j=1, ..., J, and $L(\Omega)$ be as in 2.2. Define

$$W(\Omega) = (H^m(\Omega))^N,\ W_0(\Omega) = (H_0^m(\Omega))^N,\ X(\Omega) = \prod_{j=1}^{J} H^{m_j}(\Omega),\ X_0(\Omega) = \prod_{j=1}^{J} H_0^{m_j}(\Omega)\ ,$$

these spaces being endowed with Hilbert norms as in Lemma I.4.5.

Let B and $B(x,\xi)$ be, respectively, the operator from $W(\Omega)$ into $X(\Omega)$ and the matrix J×N as defined in 2.3, except that here the coefficients $b^{\alpha}_{j,p}\in C^{m_j}_{\emptyset}(\Omega)$, the summation defining $B_{j,p}$ runs over $|\alpha|_1\le m-m_j$ ($\alpha\in\mathbb{N}_0^n$), and (x,ξ) in $B(x,\xi)$ — instead of simply $B(\xi)$ — ranges over $\overline{\Omega}\times\mathbb{C}^n$.

The sesquilinear form a in $W(\Omega)$ is defined as in 2.4, with the exception that here the summation for $|\alpha|_1$ and $|\beta|_1$ runs over $|\alpha|_1,|\beta|_1\le m$ ($\alpha,\beta\in\mathbb{N}_0^n$); $a^{\alpha,\beta}_{p,q}\in L_\infty(\Omega)$; and, additionally, $a^{\alpha,\beta}_{p,q}\in C_{\emptyset}(\Omega)$ if $|\alpha|_1=|\beta|_1=m$. Instead of $A(\xi)$ of 2.4 we have now the matrix $A(x,\xi)$ defined, in the same way, for $(x,\xi)\in\overline{\Omega}\times\mathbb{R}^n$.

Let W_1 be a closed subspace of $W(\Omega)$ containing $W_0(\Omega)$ and define

$$V_1 := \{u \in W_1 : Bu=0\}.$$

Instead of (2.1) and (2.2) assume that

(i) $\forall (x,\xi) \in \overline{\Omega} \times \mathbb{R}^n \backslash \{0\}$, rank $B(x,\xi) = J$;

(ii) $\forall (x,\xi) \in \partial\Omega \times \mathbb{C}^n \backslash \{0\}$, rank $B(x,\xi) = J$;

(iii) $\exists c_o > 0, \lambda_o, \tau_o \in \mathbb{R} : \forall u \in W_1$,

$$c_o \|u\|^2_{W(\Omega)} \leq a(u,u) + \tau_o \|Bu\|^2_{X(\Omega)} + \lambda_o \|u\|^2_{L(\Omega)} .$$

Let L_1 be the closure of V_1 in $L(\Omega)$. The preceding hypotheses ensure then that (V_1,L_1,a) is a variational triplet satisfying Proposition I.4.11, and the corresponding associated operator in L_1 has for counting function

$$N(\lambda, V_1, L(\Omega), a) ,$$

which we shall abbreviate to $N(\lambda)$ in this section.

1.2. In the particular case

$N=n\geq 2$, $J=m=1$, $m_1=0$;

$a^{\alpha,\beta}_{p,q} = 1$ if $q=p \wedge \alpha=\beta\neq 0$, 0 otherwise;

$b^{\alpha}_{1,p} = 1$ if $\alpha:=(\alpha_j)^N_{j=1}\neq 0 \wedge \alpha_p=1$, 0 otherwise;

$W_1=W_o(\Omega)$;

the above procedure gives the Stokes operator.

1.3. In order to describe the main result of [Métivier 1978], for $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^n \backslash \{0\}$ define

$V_{x,\xi} = \operatorname{Ker} B(x,\xi) \subset \mathbb{C}^N$;

$a_{x,\xi}(u,v) = A(x,\xi)u.v$ for $u,v \in V_{x,\xi}$;

$\nu(x,\xi) = N(1, V_{x,\xi}, \mathbb{C}^N, a_{x,\xi})$.

We have then

1.4. **Proposition** [Métivier 1978, Th. 1]. *With the hypotheses stated in*

1.1,

$$N(\lambda) \sim \frac{\lambda^{n/(2m)}}{(2\pi)^n} \int_\Omega \int_{\mathbb{R}^n} \nu(x,\xi)\, d\xi\, dx \qquad \text{as} \quad \lambda \to \infty \ .$$

We make the following

1.5. CONJECTURE. *With the hypotheses stated in 1.1 and assuming further that* $a^{\alpha,\beta}_{p,q}$ *is constant when* $|\alpha|_1=|\beta|_1=m$, *and* $b^{\alpha}_{j,p}$ *is constant when* $|\alpha|_1=m-m_j$, *then*

$$N(\lambda) = \frac{\lambda^{n/(2m)}}{(2\pi)^n} \int_\Omega \int_{\mathbb{R}^n} \nu(x,\xi)\, d\xi\, dx + O\Big(\lambda^{\frac{n-1/(4m+1)}{2m}}\Big) \qquad \text{as} \quad \lambda \to \infty \ . \tag{1.1}$$

The guess for the *big o* term comes from the fact that we have performed the computations once. However, these — which can well spread over 60 pages —, apart from section 2 below (which would be part of the proof), would need to be checked in order to state the conjecture above as a proposition. The idea for the proof is to go through the proof of Métivier [1978] for Proposition 1.4, trying to optimize estimates. We do not believe that this method gives the best result, because the comparison between the actual problem and the periodic case is performed not through a tesselation but, instead, through a partition of unity. The disadvantage of this is that we have to exert a tight control when defining this partition of unity, in order to minimize the contribution coming from the intersections. In any case, some information concerning the second term in the asymptotic expansion for $N(\lambda)$ seems to be possible to attain by this method, as suggested by the conjecture. The presence of the 4m in the big o seems, moreover, to be due to the contribution of the intersections just mentioned, which cannot be annihilated with this technique. This also suggests that (1.1) might be true with 0 instead of 4m inside the big o. Indeed, for the Stokes operator and when Ω has an infinitely smooth boundary, this was shown to be the case by Kozhevnikov [1986].

So, what we really study in this chapter — in the following section — is the periodic counterpart of the problem stated in 1.1. We obtain the result that would be expected (see Proposition 2.11 below), and which can be thought of as a first step in the study of the more complicated problem we have described in this introduction.

2. The periodic case

2.1. Definition. *Given* $\delta>0$ *and* $d\in\mathbb{R}^n$ *we set*

$$C^\infty_\#(\overset{\circ}{B}{}^n_\infty(d,\delta/2)) := \{f|_{\overset{\circ}{B}{}^n_\infty(d,\delta/2)} : f\in C^\infty(\mathbb{R}^n) \text{ is periodic of period } \delta \text{ in each coordinate}\}$$

and, for each $m\in\mathbb{N}_0$, $H^m_\#(\overset{\circ}{B}{}^n_\infty(d,\delta/2))$ *is defined as the closure of* $C^\infty_\#(\overset{\circ}{B}{}^n_\infty(d,\delta/2))$ *in* $H^m(\overset{\circ}{B}{}^n_\infty(d,\delta/2))$.

Of course, $H^0_\#(\overset{\circ}{B}{}^n_\infty(d,\delta/2)) = L_2(\overset{\circ}{B}{}^n_\infty(d,\delta/2))$.

2.2. Let $N,J,m\in\mathbb{N}$ be such that $N>J$, and $m_j\in\{0,\dots,m-1\}$, $j=1,\dots,J$. Let Ω be a cube $\overset{\circ}{B}{}^n_\infty(d,\delta/2)$ with centre $d\in\mathbb{R}^n$ and side length $\delta>0$ as considered above. Denote

$$L(\Omega) := (L_2(\Omega))^N,\quad W_\#(\Omega) := (H^m_\#(\Omega))^N,\quad X_\#(\Omega) := \prod_{j=1}^{J} H^{m_j}_\#(\Omega),$$

where these spaces are endowed with the Hilbert norms as in Lemma I.4.5.

2.3. Let $B_{j,p}$, $j=1,\dots,J$, $p=1,\dots,N$, be the differential operators with constant coefficients given by

$$B_{j,p} = \sum_{|\alpha|_1=m-m_j} b^\alpha_{j,p} D^\alpha$$

(obviously, here and below α is a multi-index, that is, $\alpha\in\mathbb{N}^n_0$; the same remark applies for β below).

Define the continuous linear map B from $W_\#(\Omega)$ into $X_\#(\Omega)$ by

$$B(u_1,\dots,u_N) = \Big(\sum_{p=1}^{N} B_{j,p}u_p\Big)_{j=1,\dots,J}.$$

Whenever we write $B(\xi)$, for $\xi\in\mathbb{R}^n$, we mean the matrix with J rows and N columns, the coefficients of which are

$$b_{j,p}(\xi) = \sum_{|\alpha|_1=m-m_j} i^{m-m_j}\, b_{j,p}^{\alpha}\, \xi^{\alpha}, \qquad j=1,\dots,J;\ p=1,\dots,N.$$

($B(\xi)$ should not be confused with the operator B: whenever we write B alone, we mean the operator and not the matrix $B(\xi)$).

We shall assume that

$$\forall \xi\in\mathbb{R}^n\setminus\{0\},\ \operatorname{rank} B(\xi) = J. \tag{2.1}$$

2.4. Let a be the sesquilinear form in $W_{\#}(\Omega)$ given by

$$a(u,v) = \sum_{p,q=1}^{N} a_{p,q}(u_p,v_q), \qquad u=(u_p)_p,\ v=(v_q)_q \in W_{\#}(\Omega),$$

where

$$a_{p,q}(u_p,v_q) = \sum_{|\alpha|_1=|\beta|_1=m} \int_\Omega a_{p,q}^{\alpha,\beta}.D^\alpha u_p(x).\overline{D^\beta v_q(x)}\, dx,$$

the $a_{p,q}^{\alpha,\beta}$ inside the integral being constants satisfying the relation $a_{p,q}^{\alpha,\beta} = \overline{a_{q,p}^{\beta,\alpha}}$.

Thus a is a Hermitian continuous form in $W_{\#}(\Omega)$.

By $A(\xi)$, $\xi\in\mathbb{R}^n$, we mean the $N\times N$ Hermitian matrix with coefficients

$$A_{p,q}(\xi) = \sum_{|\alpha|_1=|\beta|_1=m} a_{p,q}^{\alpha,\beta}.\xi^{\alpha+\beta}, \qquad p,q=1,\dots,N,$$

where here p denotes columns and q denotes rows.

We assume that there exists $\tau\in\mathbb{R}$ such that

$$\forall \xi\in\mathbb{R}^n\setminus\{0\},\ A(\xi)+\tau B^*(\xi)B(\xi) \text{ is positive definite}, \tag{2.2}$$

where $B^*(\xi)$ denotes the matrix adjoint to $B(\xi)$.

This of course implies (due to continuity) that there exists $c_0>0$ such that

$$\forall \xi\in\mathbb{R}^n, u\in\mathbb{C}^N, \qquad |\xi|_2=1, |u|_2=1 \Rightarrow (A(\xi)+\tau B^*(\xi)B(\xi))u.u \ge c_0. \tag{2.3}$$

Note also that, given $\rho\in\mathbb{R}$, $\xi\in\mathbb{R}^n$,

$$A(\rho\xi) = \rho^{2m}A(\xi), \tag{2.4}$$

and this and (2.3) imply that

$$\forall \xi \in \mathbb{R}^n, u \in \mathbb{C}^N, \quad B(\xi)u = 0 \Rightarrow A(\xi)u.u \geq c_o |\xi|_2^{2m} |u|_2^2 . \tag{2.5}$$

2.5. We recall that in this section Ω is a cube with centre $d \in \mathbb{R}^n$ and side length $\delta > 0$ — see 2.2 above. We further assume from now on that $\delta \leq 2\pi$.

Note that any $u = (u_1, \ldots, u_N) \in L(\Omega)$ can be written as

$$u = \Big(\sum_{\nu \in 2\pi\delta^{-1}\mathbb{Z}^n} U_j(\nu).\varphi_\nu \Big)_{j=1,\ldots,N} = \sum_{\nu \in 2\pi\delta^{-1}\mathbb{Z}^n} \varphi_\nu(\cdot)\,(U_j(\nu))_{j=1,\ldots,N} \qquad \text{in } L(\Omega),$$

where $U_j(\nu) := \int_\Omega u_j(x)\overline{\varphi_\nu(x)}\,dx$, $j=1, \ldots, N$, and $(\varphi_\nu)_{\nu \in 2\pi\delta^{-1}\mathbb{Z}^n} := (\delta^{-n/2} e^{i\nu.(\cdot - d)})_{\nu \in 2\pi\delta^{-1}\mathbb{Z}^n}$ is a complete orthonormal system in $L_2(\Omega)$.

We have then, by Parseval's formula,

$$\|u\|^2_{L(\Omega)} = \sum_{j=1}^N \|u_j\|^2_{L_2(\Omega)} = \sum_{j=1}^N \sum_{\nu \in 2\pi\delta^{-1}\mathbb{Z}^n} |U_j(\nu)|^2 = \sum_{\nu \in 2\pi\delta^{-1}\mathbb{Z}^n} |U(\nu)|_2^2 , \tag{2.6}$$

where $U(\nu) := (U_1(\nu), \ldots, U_N(\nu)) \in \mathbb{C}^N$.

If, moreover, $u \in W_\#(\Omega)$ then each u_j is in $H^m_\#(\Omega)$ and therefore

$$\|u\|^2_{W_\#(\Omega)} = \sum_{j=1}^N \sum_{|\alpha|_1 \leq m} \sum_{\nu \in 2\pi\delta^{-1}\mathbb{Z}^n} |i^{|\alpha|_1} \nu^\alpha U_j(\nu)|^2 \leq C' \sum_{\nu \in 2\pi\delta^{-1}\mathbb{Z}^n} (1 + |\nu|_2^{2m}) |U(\nu)|_2^2 , \tag{2.7}$$

where C' depends only on m and n.

We can also write, for $u \in W_\#(\Omega)$,

$$\begin{aligned} a(u,u) &= \sum_{p,q=1}^N \sum_{|\alpha|_1 = |\beta|_1 = m} a^{\alpha,\beta}_{p,q} . (D^\alpha u_p, D^\beta u_q)_{L_2(\Omega)} \\ &= \sum_{p,q=1}^N \sum_{|\alpha|_1 = |\beta|_1 = m} a^{\alpha,\beta}_{p,q} \sum_{\nu \in 2\pi\delta^{-1}\mathbb{Z}^n} i^{|\alpha|_1} (-i)^{|\beta|_1} \nu^{\alpha+\beta} U_p(\nu) \overline{U_q(\nu)} \\ &= \sum_{\nu \in 2\pi\delta^{-1}\mathbb{Z}^n} \sum_{q=1}^N \Big(\sum_{p=1}^N A_{p,q}(\nu) U_p(\nu) \Big) \overline{U_q(\nu)} \\ &= \sum_{\nu \in 2\pi\delta^{-1}\mathbb{Z}^n} A(\nu)U(\nu).U(\nu) \end{aligned} \tag{2.8}$$

and

$$Bu = \Big(\sum_{p=1}^N \sum_{|\alpha|_1 = m - m_j} b^\alpha_{j,p} \sum_{\nu \in 2\pi\delta^{-1}\mathbb{Z}^n} i^{|\alpha|_1} \nu^\alpha U_p(\nu)\, \varphi_\nu \Big)_{j=1,\ldots,J}$$

$$= \Big(\sum_{\nu\in 2\pi\delta^{-1}\mathbb{Z}^n} \Big(\sum_{p=1}^{N} b_{j,p}(\nu)\, U_p(\nu) \Big) \varphi_\nu \Big)_{j=1,\dots,J} \tag{2.9}$$

$$= \sum_{\nu\in 2\pi\delta^{-1}\mathbb{Z}^n} \varphi_\nu(\cdot)\, B(\nu)U(v) \qquad \text{in} \quad \prod_{j=1}^{J} L_2(\Omega)\,.$$

Formulae (2.6) to (2.9) together with (2.5) prove that

$$\forall u\in W_\#(\Omega), \qquad Bu=0 \Rightarrow a(u,u)+c_0\|u\|^2_{L(\Omega)} \geq c_0 C'^{-1}\|u\|^2_{W_\#(\Omega)}\ ,$$

which means that a is coercive on $V_\#(\Omega) := \{u\in W_\#(\Omega) : Bu=0\}$ and, consequently, $(V_\#(\Omega),L(\Omega),a)$ *is a variational triplet* — cf. I.4.1.

2.6. LEMMA [Métivier 1978, Lem. 6.3]. *If* $\lambda>0$ *and* $\delta\in(0,2\pi]$

$$N(\lambda,V_\#(\Omega),L(\Omega),a) = N + \sum_{\substack{\nu\in 2\pi\delta^{-1}\mathbb{Z}^n \\ \nu\neq 0}} N(\lambda,V_\nu,\mathbb{C}^N,a_\nu)\ ,$$

where $V_\xi := \operatorname{Ker} B(\xi)$ *and* $a_\xi(u,v) := A(\xi)u.v$ *for all* $u,v\in V_\xi$ *and* $\xi\in\mathbb{R}^n$.

2.7. Clearly, for each $\xi\in\mathbb{R}^n\setminus\{0\}$, $N(\lambda,V_\xi,\mathbb{C}^N,a_\xi) = N(\lambda,V_\xi,V_\xi,a_\xi)$, the number of eigenvalues (repeated according to multiplicity) of the associated operator acting on V_ξ that are less than or equal to λ. On account of (2.1), $\dim V_\xi = N-J$, therefore that operator has precisely $N-J$ eigenvalues, which we shall denote by $\lambda_k(\xi)$, $k=1,\dots,N-J$, (arranged in non-decreasing order). Note also that, due to (2.5), all the $\lambda_k(\xi)$ are strictly positive.

From the definition of $B(\xi)$ it follows that $V_{\rho\xi} = V_\xi$ for $\rho>0$, and, from (2.4), $\lambda_k(\rho\xi) = \rho^{2m}\lambda_k(\xi)$, $k=1,\dots,N-J$; that is, for each $k\in\{1,\dots,N-J\}$

$$\begin{array}{rcl} \mathbb{R}^n\setminus\{0\} & \longrightarrow & (0,\infty) \\ \xi & \longmapsto & \lambda_k(\xi) \end{array} \quad \text{is homogeneous of degree } 2m\,. \tag{2.10}$$

It is convenient to give the following characterization of the $\lambda_k(\xi)$ — see [Métivier 1978, proof of Lem. 6.2]: $\lambda_k(\xi)$, $k=1,\dots,N-J$, *are the non-vanishing eigenvalues of* $P(\xi)A(\xi)P(\xi)$, *where* $P(\xi)$ *is the* $N\times N$ *matrix of the orthogonal projection of* $\mathbb{C}^N$ *onto* V_ξ; more precisely, denoting by $\lambda_k(P(\xi)A(\xi)P(\xi))$, $k=1,\dots,N$, the eigenvalues of $P(\xi)A(\xi)P(\xi)$ indexed in non-decreasing order, the following holds:

$$\lambda_k(\xi) = \lambda_{J+k}(P(\xi)A(\xi)P(\xi)), \quad \xi\neq 0, \ k\in\{1,\ldots,N-J\}; \tag{2.11}$$

moreover, $P(\xi) = \mathrm{Id} - B^*(\xi)(B(\xi)B^*(\xi))^{-1}B(\xi)$, *where* Id *is the identity matrix on* $\mathbb{C}^N$ (note that $(B(\xi)B^*(\xi))^{-1}$ is well-defined because of (2.1)).

2.8. From now on we shall denote more simply $N(\lambda,\xi) := N(\lambda,V_\xi,\mathbb{C}^N,a_\xi)$.

A consequence of (2.10) is the identity

$$N(\lambda,\xi) = N(\rho^{2m}\lambda,\rho\xi), \qquad \rho>0, \ \xi\neq 0,$$

so that, from Lemma 2.6 and with $\lambda>0$, $\delta\in(0,2\pi]$, $\mu := 2\pi\delta^{-1}\lambda^{-1/(2m)}$,

$$\begin{aligned} N(\lambda,V_\#(\Omega),L(\Omega),a) &= N + \sum_{\substack{\nu\in\mu\mathbb{Z}^n\\ \nu\neq 0}} N(1,\nu) \\ &= N + \lambda^{n/(2m)}\left(\frac{\delta}{2\pi}\right)^n \int_{\mathbb{R}^n} \sum_{\substack{\nu\in\mu\mathbb{Z}^n\\ \nu\neq 0}} N(1,\nu).\mathbb{1}_{\overset{\circ}{B}{}^n_\infty(\nu,\mu/2)}(\xi)\, d\xi . \end{aligned} \tag{2.12}$$

It was with the help of this relation that Métivier [1978, Prop. 6] proved that, for $\delta\in(0,2\pi]$,

$$N(\lambda,V_\#(\Omega),L(\Omega),a) \sim \lambda^{n/(2m)}\left(\frac{\delta}{2\pi}\right)^n \int_{\mathbb{R}^n} N(1,\xi)\, d\xi \qquad \text{as } \lambda\to\infty . \tag{2.13}$$

Here we are interested in estimating the remainder of this asymptotic formula.

We begin with a perturbation result:

2.9. LEMMA. *For each* $k\in\{1,\ldots,N-J\}$, $\lambda_k(\cdot)$ *is a Lipschitz-continuous function on each compact subset of* $\mathbb{R}^n\setminus\{0\}$.

Proof. We recall the characterization (2.11) of $\lambda_k(\xi)$ and note that $T(\xi) := P(\xi)A(\xi)P(\xi)$ is a positive semidefinite Hermitian matrix on a Hilbert space. This implies that its eigenvalues are its s-numbers. Of course, since the s-numbers are usually listed not in non-decreasing but in non-increasing order, we just have to be a bit careful when expressing the identification of the s-numbers $s_k(T(\xi))$ with the eigenvalues $\lambda_k(T(\xi))$, $k\in\{1,\ldots,N\}$:

$$\lambda_k(T(\xi)) = s_{N-k+1}(T(\xi)), \qquad k=1,\dots,N,$$

so that, by (2.11),

$$\lambda_k(\xi) = s_{N-J-k+1}(T(\xi)), \qquad k=1,\dots,N-J. \tag{2.14}$$

However, the important thing to note here is that once one of the indices in (2.14) is fixed, the other is fixed too, and we have, by a well-known property of s-numbers (see, for example, [Pietsch 1980, 11.1.3]),

$$\begin{aligned} |\lambda_k(\xi) - \lambda_k(\zeta)| &\le \|T(\xi) - T(\zeta)\| \\ &\le \Big(\sum_{p=1}^{N}\sum_{q=1}^{N} |t_{p,q}(\xi) - t_{p,q}(\zeta)|^2\Big)^{1/2}, \qquad \xi,\zeta\in\mathbb{R}^n\backslash\{0\},\ k=1,\dots,N-J, \end{aligned} \tag{2.15}$$

where $t_{p,q}(\xi)$ is the coefficient of $T(\xi)$ in row p and column q, $p,q=1,\dots,N$. Accordingly, the lemma will be proved if we show that each function $t_{p,q}(\cdot)$ is Lipschitz-continuous on each compact subset of $\mathbb{R}^n\backslash\{0\}$, or, what is enough (by the mean value theorem), if each of those functions is of class C^1 on $\mathbb{R}^n\backslash\{0\}$.

Let us then analyse how the matrix $T(\xi) = P(\xi)A(\xi)P(\xi)$ is obtained. By the characterization of $P(\xi)$ given towards the end of 2.7, we have

$$T(\xi) = (Id - B^*(\xi)(B(\xi)B^*(\xi))^{-1}B(\xi))\,A(\xi)\,(Id - B^*(\xi)(B(\xi)B^*(\xi))^{-1}B(\xi)). \tag{2.16}$$

Note that:

(1) the coefficients of Id, $A(\xi)$, $B(\xi)$, $B^*(\xi)$ and $B(\xi)B^*(\xi)$ are polynomials in ξ, hence of class C^1;

(2) following the theoretical construction of the inverse of a matrix (see [Lang 1965, p. 334] for example), we get that the coefficients of $(B(\xi)B^*(\xi))^{-1}$ are quotients of polynomials in ξ with common denominator given by $\det(B(\xi)B^*(\xi))$, which is different from zero because of (2.1); therefore the coefficients of $(B(\xi)B^*(\xi))^{-1}$ are also of class C^1;

(3) finally and according to (2.16), for any $p,q\in\{1,\dots,N\}$ $t_{p,q}(\xi)$ is obtained

by a finite combination of sums and products of the coefficients of the matrices described in (1) and (2), hence it is also of class C^1, as required.

2.10. LEMMA. *There are positive constants* c_1 *and* c_2 *such that for all* $k\in\{1,\dots,N-J\}$

$$\mathrm{Vol}_n(\Lambda(k)_\varepsilon) \le c_1\varepsilon \quad \text{whenever} \quad \varepsilon\in]0,c_2[\,,$$

where $\Lambda(k) := \{\xi\in\mathbb{R}^n\setminus\{0\} : \lambda_k(\xi)=1\}$ *and* $\Lambda(k)_\varepsilon$ *is its* ε-neighbourhood, *that is,* $\Lambda(k)_\varepsilon = \{\xi\in\mathbb{R}^n : \mathrm{dist}\,(\xi,\Lambda(k)) < \varepsilon\}$.

Proof. For each k let f_k be the map

$$f_k : \begin{array}{l} S^{n-1} \longrightarrow \Lambda(k) \\ \xi \longmapsto \lambda_k(\xi)^{-1/(2m)}\xi \end{array} \quad ,$$

which, as is easily seen (with the help of (2.10), it is well-defined and is a bijection.

It is also Lipschitz-continuous:

$$\begin{aligned} |f_k(\zeta)-f_k(\xi)|_2 &\le \lambda_k(\zeta)^{-1/(2m)}|\zeta-\xi|_2 + |\lambda_k(\zeta)^{-1/(2m)}-\lambda_k(\xi)^{-1/(2m)}||\xi|_2 \\ &\le c_o^{-1/(2m)}|\zeta-\xi|_2 + \frac{c_o^{-1/(2m)-1}}{2m}|\lambda_k(\zeta)-\lambda_k(\xi)| \qquad (2.17) \\ &\le c_o^{-1/(2m)}\Big(1+\frac{C(k)}{2mc_o}\Big)|\zeta-\xi|_2 \,, \end{aligned}$$

taking account of (2.5) — the c_o here is the same as over there —, the mean value theorem and Lemma 2.9 (C(k) is a Lipschitz constant corresponding to the compact S^{n-1} and the function $\lambda_k(\cdot)$).

Put

$$C'' := \max_{k=1,\dots,N-J} c_o^{-1/(2m)}\Big(1+\frac{C(k)}{2mc_o}\Big)\,. \qquad (2.18)$$

In the proof of (2.17) above we used the fact that (2.5) implies the relation

$$\lambda_k(\xi) \ge c_o|\xi|_2^{2m} \qquad \text{for} \quad \xi\neq 0\,. \qquad (2.19)$$

It will be helpful to have an inequality in the opposite direction, and this indeed holds:

$$\lambda_k(\xi) = A(\xi)u.u \le \|A(\xi)\| \le M|\xi|_2^{2m}, \qquad \xi\ne 0, \tag{2.20}$$

where here u is some normalized eigenvector corresponding to $\lambda_k(\xi)$, and M depends only on the $a_{p,q}^{\alpha,\beta}$, $|\alpha|_1=|\beta|_1=m$, p,q=1, ..., N, given in 2.4.

Let $0<\varepsilon<M^{-1/(2m)}$ and consider $\xi\in\Lambda(k)_\varepsilon$. Then there exists $\zeta\in\Lambda(k)$ such that $|\xi-\zeta|_2<\varepsilon$ and, due to (2.20) and the choice of ε, $\xi\ne 0$. Furthermore,

$$\begin{aligned}|\xi - f_k(|\xi|_2^{-1}\xi)|_2 &\le |\xi-\zeta|_2 + |\zeta - f_k(|\xi|_2^{-1}\xi)|_2 < \varepsilon + |f_k(|\zeta|_2^{-1}\zeta) - f_k(|\xi|_2^{-1}\xi)|_2 \\ &\le \varepsilon + C''\,\big||\zeta|_2^{-1}\zeta - |\xi|_2^{-1}\xi\big|_2 \le \varepsilon + C''\frac{|\zeta-\xi|_2 + \big||\zeta|_2 - |\xi|_2\big|}{|\zeta|_2} \\ &< \varepsilon + 2C''M^{1/(2m)}\varepsilon \le (1+2C''M^{1/(2m)})M^{1/(2m)}\lambda_k(\xi)^{-1/(2m)}|\xi|_2\,\varepsilon,\end{aligned} \tag{2.21}$$

where we used (2.17), with C" given by (2.18), and (2.20). Denote

$$C := (1+2C''M^{1/(2m)})M^{1/(2m)}, \qquad c_2 := C^{-1} \tag{2.22}$$

and assume that $\varepsilon\in]0,c_2[$. Since $f_k(|\xi|_2^{-1}\xi) = \lambda_k(\xi)^{-1/(2m)}\xi$, from (2.21) it follows

$$|1-\lambda_k(\xi)^{-1/(2m)}| < C\varepsilon\,\lambda_k(\xi)^{-1/(2m)},$$

that is,

$$(1-C\varepsilon)^{2m} < \lambda_k(\xi) < (1+C\varepsilon)^{2m} \qquad \text{for } \xi\in\Lambda(k)_\varepsilon \text{ and } \varepsilon\in]0,c_2[\,. \tag{2.23}$$

Now define $D(k) := \{\xi\in\mathbb{R}^n\setminus\{0\} : \lambda_k(\xi)\le 1\}$ and note that, due to homogeneity (cf. (2.10)),

$$\rho D(k) = \{\xi\in\mathbb{R}^n\setminus\{0\} : \lambda_k(\xi)\le\rho^{2m}\} \qquad \text{for all } \rho>0\,.$$

This and (2.23) then yields $\Lambda(k)_\varepsilon \subset (1+C\varepsilon)D(k)\setminus(1-C\varepsilon)D(k)$ and

$$\begin{aligned}\mathrm{Vol}_n(\Lambda(k)_\varepsilon) &\le \mathrm{Vol}_n((1+C\varepsilon)D(k)) - \mathrm{Vol}_n((1-C\varepsilon)D(k)) \\ &\le ((1+C\varepsilon)^n - (1-C\varepsilon)^n)\,\mathrm{Vol}_n(D(k))\end{aligned}$$

$$\leq n2^n C\varepsilon \mathrm{Vol}_n(D(k)) \leq n2^n C c_0^{-n/(2m)} \mathrm{Vol}_n(B_2^n).\varepsilon\,, \quad \text{for } \varepsilon\in]0,c_2[\,,$$

using the mean value theorem and (2.19).

The proof of the lemma will be complete after we choose $c_1 := n2^n C c_0^{-n/(2m)} \mathrm{Vol}_n(B_2^n)$, with C given by (2.22) and c_0 as in (2.3).

2.11. **Proposition.** *With the hypotheses set forth in 2.2 to 2.4 and the beginning of 2.5, with* $V_{\#}(\Omega)$ *as defined towards the end of 2.5,* V_ξ *and* a_ξ *as in Lemma 2.6 and, as in 2.8, using* $N(\lambda,\xi)$ *as a shorthand for* $N(\lambda,V_\xi,\mathbb{C}^N,a_\xi)$*, there are positive constants* c_3 *and* c_4 *such that, if* $\lambda > c_3\delta^{-2m}$,

$$\left|N(\lambda,V_{\#}(\Omega),L(\Omega),a) - \lambda^{n/(2m)}\left(\frac{\delta}{2\pi}\right)^n \int_{\mathbb{R}^n} N(1,\xi)\, d\xi\right| \leq c_4\delta^{n-1}\lambda^{(n-1)/(2m)}$$

(note that we mean that c_3 *and* c_4 *are independent of* λ*,* d *and* δ*).*

Proof. From (2.12) we have, for $\lambda>0$ and with $\mu = 2\pi\delta^{-1}\lambda^{-1/(2m)}$,

$$\begin{aligned}&\left|N(\lambda,V_{\#}(\Omega),L(\Omega),a) - \lambda^{n/(2m)}\left(\frac{\delta}{2\pi}\right)^n \int_{\mathbb{R}^n} N(1,\xi)\, d\xi\right| \leq \\ &\leq N + \lambda^{n/(2m)}\left(\frac{\delta}{2\pi}\right)^n \left|\int_{\mathbb{R}^n} N(1,\xi) - \sum_{\substack{\nu\in\mu\mathbb{Z}^n\\ \nu\neq 0}} N(1,\nu).\mathbb{1}_{\overset{o}{B}{}^n_\infty(\nu,\mu/2)}(\xi)\, d\xi\right|.\end{aligned} \tag{2.24}$$

The latter integral can be written as

$$\int_{\overset{o}{B}{}^n_\infty(0,\mu/2)} N(1,\xi)\, d\xi + \int_{\bigcup_{\substack{\nu\in\mu\mathbb{Z}^n\\ \nu\neq 0}} \overset{o}{B}{}^n_\infty(\nu,\mu/2)} N(1,\xi) - N(1,\nu(\xi))\, d\xi\,, \tag{2.25}$$

where $\nu(\xi)\in\mu\mathbb{Z}^n$ is such that $\xi\in\overset{o}{B}{}^n_\infty(\nu(\xi),\mu/2)$.

The first integral in (2.25) can obviously be estimated from above by

$$(N-J)\,\mathrm{Vol}_n(\overset{o}{B}{}^n_\infty(0,\mu/2)) = (N-J)\left(\frac{2\pi}{\delta}\right)^n \lambda^{-n/(2m)}.$$

We write the second integral in (2.25) as

$$\int_{\bigcup_{\substack{\nu\in\mu\mathbb{Z}^n\\ \nu\neq 0}} \overset{o}{B}{}^n_\infty(\nu,\mu/2) \cap \bigcup_{k=1}^{N-J} \Lambda(k)_{\sqrt{n}\mu/2}} N(1,\xi) - N(1,\nu(\xi))\, d\xi + \int_{\bigcup_{\substack{\nu\in\mu\mathbb{Z}^n\\ \nu\neq 0}} \overset{o}{B}{}^n_\infty(\nu,\mu/2) \setminus \bigcup_{k=1}^{N-J} \Lambda(k)_{\sqrt{n}\mu/2}} N(1,\xi) - N(1,\nu(\xi))\, d\xi\,. \tag{2.26}$$

In the second integral in (2.26) $\operatorname{dist}(\xi,\Lambda(k)) \geq \sqrt{n}\mu/2$ for all $k\in\{1,\dots,N-J\}$, so that $\lambda_k(\xi) \neq 1$ and, since $\xi \in \mathring{B}^n_\infty(\nu(\xi),\mu/2)$, $\operatorname{dist}(\nu(\xi),\Lambda(k)) \geq \operatorname{dist}(\xi,\Lambda(k)) - |\xi-\nu(\xi)|_2 > \sqrt{n}\mu/2 - \sqrt{n}\mu/2 = 0$, that is, $\lambda_k(\nu(\xi)) \neq 1$ too. Of course, for each $k\in\{1,\dots,N-J\}$ it must either be $\lambda_k(\xi), \lambda_k(\nu(\xi)) < 1$ or both >1, for otherwise, due to the continuity of $\lambda_k(\cdot)$, there would exist $\zeta = t\xi+(1-t)\nu(\xi)$, for some $t\in]0,1[$, such that $\lambda_k(\zeta) = 1$, that is, $\zeta\in\Lambda(k)$ and $|\xi-\zeta|_2 \leq |\xi-\nu(\xi)|_2 < \sqrt{n}\mu/2$, which contradicts the fact that $\operatorname{dist}(\xi,\Lambda(k)) \geq \sqrt{n}\mu/2$. Hence the integrand in the second integral in (2.26) is zero.

The absolute value of the first integral in (2.26) can clearly be estimated from above by

$$(N-J)\sum_{k=1}^{N} \operatorname{Vol}_n(\Lambda(k)_{\sqrt{n}\mu/2}) \leq (N-J)^2 c_1 \sqrt{n}\mu/2 \qquad \text{if } \sqrt{n}\mu/2 < c_2,$$

where c_1 and c_2 are as in Lemma 2.10.

Putting all this together in (2.24) we obtain, for $\lambda > \left(\frac{\sqrt{n}\pi}{c_2}\right)^{2m}\delta^{-2m}$ $(= (\sqrt{n}\pi C)^{2m}\delta^{-2m}$, by (2.22)),

$$\left|N(\lambda,V_{\#}(\Omega),L(\Omega),a) - \lambda^{n/(2m)}\left(\frac{\delta}{2\pi}\right)^n\int_{\mathbb{R}^n} N(1,\xi)\,d\xi\right| \leq$$

$$\leq N + \lambda^{n/(2m)}\left(\frac{\delta}{2\pi}\right)^n\left((N-J)\left(\frac{2\pi}{\delta}\right)^n\lambda^{-n/(2m)} + (N-J)^2 c_1\sqrt{n}\,\pi\delta^{-1}\lambda^{-1/(2m)}\right)$$

$$= N + N - J + (N-J)^2 c_1\sqrt{n}\,2^{-n}\pi^{1-n}\delta^{n-1}\lambda^{(n-1)/(2m)}$$

$$\leq \left((2N-J)(\sqrt{n}\pi C)^{1-n} + (N-J)^2 n^{3/2} C c_0^{-n/(2m)} \operatorname{Vol}_n(B^n_2)\,\pi^{1-n}\right)\delta^{n-1}\lambda^{(n-1)/(2m)},$$

with C and c_0 respectively as in (2.22) and (2.3), this finishing our proof.

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