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Homogeneity, non-smooth atoms and Besov spaces of generalised smoothness on quasi-metric spaces

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Abstract

An h-space is a compact set with respect to a quasi-metric and endowed with a Borel measure such that the measure of a ball of radius r is equivalent to $h(r)$, for some function h. Applying an approach introduced by Triebel in [28] we define Besov spaces of generalised smoothness on h-spaces. We describe the techniques and tools used in this construction, namely snowflaked transforms and charts. This approach relies on using what is known for function spaces on some fractal sets, which are themselves defined as traces of convenient function spaces on $\mathbb{R}^n$. It revealed to be important to obtain new properties and characterisations for the elements of these spaces, for example, to guarantee the independence of the charts used. So we also present results for Besov spaces of generalised smoothness on $\mathbb{R}^n$ and some special fractal sets, namely characterisations by differences and a homogeneity property (on $\mathbb{R}^n$) and non-smooth atomic decompositions.

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1. Introduction

An *h-space* $X = (X, \varrho, \mu)$ is a compact set X with respect to a quasi-metric ϱ and endowed with a Borel measure μ such that the measure of a ball of radius r is equivalent to $h(r)$, for some function h, i.e.,

$$\mu(B(x,r)) \sim h(r) \quad \text{for all } x \in X \text{ and } 0 < r \leq \operatorname{Diam} X. \tag{1.0.1}$$

If we have in particular $X \subset \mathbb{R}^n$ and ϱ the usual Euclidean metric, we use a different notation: we denote the set by the letter Γ and the metric by ϱ_n. We say in this case that $\Gamma = (\Gamma, \varrho_n, \mu)$ is an *h-set*. In [3] and [5] Bricchi characterised the class of admitted functions h. There he also introduced and studied Besov spaces of generalised smoothness on these fractal sets. We also refer to [17] for characterisations of these function spaces, which will be used later in this paper. In the particular case where $h(r) = r^d$, for some $d > 0$, we say that Γ is a *d-set*. Function spaces on d-sets have been studied with several methods. We refer to Jonsson and Wallin (cf. [15]) and Triebel (cf. [25] and [26]).
Another particular class of h-spaces that have been considered are *d-spaces*. In these cases, we also have $h(r) = r^d$, for some $d > 0$, but now for (abstract) quasi-metric spaces X. Function spaces on this kind of spaces have been studied by Han and Yang, namely in [12], where *approximations to the identity* are used to define the spaces. We refer to [29, 1.17.4], where several references are given and a comparison between this approach to function spaces on quasi-metric spaces and the description of spaces $B^s_{p,q}(\mathbb{R}^n)$ in terms of local means is made.

In [28] Triebel presented a different approach to define Besov spaces on d-spaces, using *snowflaked transforms* and *Euclidean charts*, which allow to transfer the study of function spaces on quasi-metric spaces to spaces on fractal sets in some $\mathbb{R}^n$. There were also presented results involving applications in function spaces on d-spaces, obtained by making use of what is known about d-sets. It was also proved that, in some cases, the Besov spaces defined this way are the same spaces as introduced by Han and Yang.

So, on the one hand there are works about Besov spaces defined on h-sets and on the other hand about Besov spaces defined on d-spaces. In the present paper we consider, following Triebel's approach, this more general class of h-spaces, which includes d-spaces, d-sets and h-sets.

The main idea is the following: if we consider an h-space $X = (X, \varrho, \mu)$, then for $0 < \varepsilon < \varepsilon_0 \leq 1$ there is a bi-Lipschitzian mapping L from the snowflaked version of X, $(X, \varrho^\varepsilon, \mu)$, into some $\mathbb{R}^n$. This means that

$$\varrho^\varepsilon(x,y) \sim |L(x) - L(y)|, \quad x, y \in X. \tag{1.0.2}$$

If we consider $\Gamma = L(X)$ and the image measure $\nu = \mu \circ L^{-1}$, then using (1.0.1) and (1.0.2) we conclude that (Γ, ϱ_n, ν) is an $h_{1/\varepsilon}$-set, where

$$h_{1/\varepsilon}(r) = h(r^{\frac{1}{\varepsilon}}), \quad r \in \mathbb{R}^+.$$

We say that $(X, \varrho, \mu; L)$, or just L, is an *Euclidean chart* of the h-space. To define function spaces on (X, ϱ, μ), we consider appropriate function spaces on the $h_{1/\varepsilon}$-set and then we use the chart to transfer everything to the h-space.
Hence in this paper we study first function spaces on $\mathbb{R}^n$, then, via traces, on h-sets, and finally, using charts, on h-spaces.
To prove the independence of the charts in the definition of these function spaces, we rely on non-smooth atomic representations for their elements. In the case of d-spaces, to obtain this kind of characterisation, Triebel used non-smooth atomic characterisations for Besov spaces on $\mathbb{R}^n$ (cf. [27]) and also two different approaches for Besov spaces on d-sets, namely the one introduced by Jonsson and Wallin (cf. [15]) and smooth atomic decompositions (cf. [25]).

A further important tool was the *homogeneity* property for spaces on $\mathbb{R}^n$ (for Triebel-Lizorkin spaces we refer to [26, Corollary 5.16, p. 66] and for Besov spaces we refer to [8]). So, considering $A \in \{B, F\}$, for all $0 < p, q \leq \infty$ (with $p < \infty$ for the F-spaces) and $s > n(1/\min(1,p) - 1)$ (if $A = B$) or $s > n(1/\min(1,p,q) - 1)$ (if $A = F$),

$$\|f(2^{-k}\cdot)|A^s_{p,q}(\mathbb{R}^n)\| \sim 2^{-k(s-\frac{n}{p})}\|f|A^s_{p,q}(\mathbb{R}^n)\|, \tag{1.0.3}$$

for all $k \in \mathbb{N}_0$ and all

$$f \in A^s_{p,q}(\mathbb{R}^n) \quad \text{with} \quad \operatorname{supp} f \subset \{x \in \mathbb{R}^n : |x| \leq 2^{-k}\}. \tag{1.0.4}$$

In this paper we prove an adapted homogeneity property for Besov spaces of generalised smoothness. We obtain a counterpart of (1.0.3) and (1.0.4) for the quasi-norms of f and $f(2^{-k}\cdot)$, but with quasi-norms in "different" spaces. This is presented in Section 3 and applied in the same Section to prove a characterisation with non-smooth atoms for Besov spaces of generalised smoothness on $\mathbb{R}^n$.

We would like to remark that for all $x, y \in X$ and $j \in \mathbb{N}_0$

$$\varrho(x,y) \sim 2^{-j} \quad \text{if and only if} \quad |L(x) - L(y)| \sim 2^{-\varepsilon j}.$$

Therefore, in order to work with the usual dyadic decompositions in X, we consider in $\mathbb{R}^n$ decompositions where we take $2^{\varepsilon j}$ instead of 2^j, $j \in \mathbb{N}_0$. This kind of decomposition is included in the class of decompositions considered by Farkas and Leopold in [11] and by Moura in [21]. We apply some results proved in these papers, namely characterisations with atomic decompositions and with differences. We also use a standardisation result proved by Caetano and Leopold (cf. [7]) to reduce these spaces to corresponding spaces where usual dyadic decompositions can be taken.

The paper is organised as follows. In Section 2 we collect some notation and concepts which usually appear in the context of generalised smoothness.

In Section 3 we present results for Besov spaces of generalised smoothness on $\mathbb{R}^n$, such as atomic decompositions, characterisations with differences and an adapted homogeneity property.

In Section 4 we collect some results for Besov spaces on h-sets and obtain characterisations with non-smooth atomic decompositions.

Finally, in Section 5 we describe the technique to introduce function spaces on quasi-metric spaces using Euclidean charts and study the (in)dependence of the charts. We also present an example where we use the same tools to get easily estimates for entropy numbers of embeddings between the mentioned spaces taking advantage of what is already known on $\mathbb{R}^n$.

During the preparation of this paper we had the opportunity to discuss the included material extensively with Professor Triebel, who also read previous versions of the manuscript. We would like to thank him for this and for the valuable suggestions given.

2. Preliminaries

2.1. General Notation. First we introduce some standard notation and useful definitions. As usual, $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{R}^n$, $n \in \mathbb{N}$, stands for the n-dimensional real Euclidean space and $\mathbb{R} = \mathbb{R}^1$. We denote by $\mathbb{Z}$ the collection of all integers; and by $\mathbb{Z}^n$, where $n \in \mathbb{N}$, the lattice of all points $m = (m_1, ..., m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. Let $\mathbb{N}_0^n$, where $n \in \mathbb{N}$, be the set of all multi-indices

$$\alpha = (\alpha_1, ..., \alpha_n) \quad \text{with} \quad \alpha_j \in \mathbb{N}_0 \quad \text{and} \quad |\alpha| = \sum_{j=1}^{n} \alpha_j.$$

We denote by $[\cdot]$ the integer-part function.
We use the symbol "$\lesssim$" in

$$a_k \lesssim b_k \quad \text{or} \quad \varphi(r) \lesssim \psi(r)$$

always to mean that there is a positive number c_1 such that

$$a_k \le c_1 b_k \quad \text{or} \quad \varphi(r) \le c_1 \psi(r)$$

for all admitted values of the discrete variable k or the continuous variable r, where $(a_k)_k$, $(b_k)_k$ are non-negative sequences and φ, ψ are non-negative functions. We use the equivalence "$\sim$" in

$$a_k \sim b_k \quad \text{or} \quad \varphi(r) \sim \psi(r)$$

for

$$a_k \lesssim b_k \quad \text{and} \quad b_k \lesssim a_k \qquad \text{or} \qquad \varphi(r) \lesssim \psi(r) \quad \text{and} \quad \psi(r) \lesssim \varphi(r).$$

We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ equipped with the usual topology, and by $\mathcal{S}'(\mathbb{R}^n)$ its topological dual, the space of all tempered distributions on $\mathbb{R}^n$.
As usual, "domain" stands for "open set". If Ω is a domain in $\mathbb{R}^n$ then $L_p(\Omega)$ denotes the collection of all complex-valued Lebesgue measurable functions in Ω such that

$$\|f|L_p(\Omega)\| := \Big(\int_\Omega |f(x)|^p dx \Big)^{\frac{1}{p}}$$

(with the usual modification if $p=\infty$) is finite.
We use the standard abbreviations

$$\overline{p} := \max(1,p) \quad \text{and} \quad n\Big(\frac{1}{p}-1\Big)_+ = n\Big(\frac{1}{\min\{1,p\}}-1\Big).$$

The Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is denoted by $\widehat{\varphi}$ or $\mathcal{F}\varphi$. As usual, $\check{\varphi}$ and $\mathcal{F}^{-1}\varphi$ stand for the inverse Fourier transform. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to $\mathcal{S}'(\mathbb{R}^n)$ in the standard way. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ we will use the notation $\varphi(D)f = \mathcal{F}^{-1}(\varphi\mathcal{F}f)$.
Furthermore, if $0<q\leq\infty$ then ℓ_q has the standard meaning.
If $(f_j)_{j\in\mathbb{N}_0}$ is a sequence of complex-valued Lebesgue measurable functions on $\mathbb{R}^n$, then

$$\|(f_j)_{j\in\mathbb{N}_0}|\ell_q(L_p)\| = \Big(\sum_{j=0}^{\infty}\|f_j|L_p\|^q\Big)^{1/q}$$

with the appropriate modification if $q=\infty$.
If there is no additional information, when we speak about "functions" we are considering complex-valued functions.

2.2. Admissible sequences and functions.

Definition 2.1. Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be a sequence of positive numbers. We say that σ is an *admissible sequence* if there are positive constants d_0, d_1 such that

$$d_0\sigma_j \leq \sigma_{j+1} \leq d_1\sigma_j, \;\; j\in\mathbb{N}_0. \tag{2.2.1}$$

Example 2.2. We introduce two particular kinds of admissible sequences that we will use throughout the paper. So, we will denote by (s) the (admissible) sequence defined by

$$(s) := (2^{js})_{j\in\mathbb{N}_0}, \quad s\in\mathbb{R}.$$

Let $\psi : (0,1] \to \mathbb{R}$ be a positive monotone function such that $\psi(2^{-2j}) \sim \psi(2^{-j})$, for all $j\in\mathbb{N}_0$. We will denote by (s,ψ) the sequences

$$(s,\psi) := (2^{js}\psi(2^{-j}))_{j\in\mathbb{N}_0}, \quad s\in\mathbb{R},$$

which are also admissible sequences.

Notation 2.3. Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ and $\beta = (\beta_j)_{j\in\mathbb{N}_0}$ be admissible sequences. We denote by σ^{-1} and $\sigma\beta$ the (admissible) sequences given by

$$\sigma^{-1} = (\sigma_j^{-1})_{j\in\mathbb{N}_0} \quad \text{and} \quad \sigma\beta = (\sigma_j\beta_j)_{j\in\mathbb{N}_0},$$

respectively.

Definition 2.4. A function $\Lambda : (0,\infty) \to (0,\infty)$ will be called an *admissible function* if it is continuous and if, for any $b>0$,

$$\Lambda(bz) \sim \Lambda(z), \quad z>0.$$

Convention 2.5. Hereafter, by N we will always denote a sequence $N = (N_j)_{j\in\mathbb{N}_0}$ of real positive numbers such that there exist two numbers $1<\lambda_0\leq\lambda_1$ with

$$\lambda_0 N_j \leq N_{j+1} \leq \lambda_1 N_j, \quad j\in\mathbb{N}_0.$$

REMARK 2.6. In particular, N is admissible and is a so-called strongly increasing sequence (cf. [11]). For such N there exists a number $l_0 \in \mathbb{N}$ such that

$$2N_j \leq N_k \text{ for any } l,k \text{ such that } j+l_0 \leq k.$$

This is true if we choose, for instance, $l_0 \in \mathbb{N}_0$ such that

$$\lambda_0^{l_0} \geq 2$$

holds. We fix such an l_0 in the following.

In the next sections it will be convenient to associate an admissible function to an admissible sequence as follows:

DEFINITION 2.7. Let σ be an admissible sequence and Λ be an admissible function. We will say that Λ is *associated* to $\{\sigma, N\}$ if $\Lambda(z) \sim \sigma_j$ for any $z \in [N_j, N_{j+1}]$, for any $j \in \mathbb{N}_0$ with equivalence constants independent of j.
If $N_j = 2^j,\ j \in \mathbb{N}_0$, we shall simply write that Λ is *associated* to σ.

The next example, which can be found in [6], guarantees that, given admissible sequences σ and N, with N according to Convention 2.5, there is an admissible function Λ associated to $\{\sigma, N\}$.

EXAMPLE 2.8. Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence. The function
$\Lambda : (0,\infty) \to (0,\infty)$ defined by

$$\Lambda(z) = \begin{cases} \dfrac{\sigma_{j+1}-\sigma_j}{N_{j+1}-N_j}(z-N_j)+\sigma_j & , z \in [N_j, N_{j+1}), j \in \mathbb{N}_0 \\ \sigma_0 & , z \in (0, N_0) \end{cases}$$

is an admissible function associated to $\{\sigma, N\}$.

In the context of generalised smoothness we need something which plays the role of the regularity index s in the case of classical smoothness. We will use the indices described below.

DEFINITION 2.9. Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence and

$$\underline{\sigma}_j := \inf_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \quad \text{and} \quad \overline{\sigma}_j := \sup_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k}, \quad j \in \mathbb{N}_0.$$

The *lower* and *upper Boyd indices* of the sequence σ are defined, respectively, by

$$\underline{s}(\sigma) := \lim_{j\to\infty} \frac{\log \underline{\sigma}_j}{j} \quad \text{and} \quad \overline{s}(\sigma) := \lim_{j\to\infty} \frac{\log \overline{\sigma}_j}{j},$$

where log denotes the logarithm to the base 2.

REMARK 2.10. For an admissible sequence σ, the sequence $(\log \overline{\sigma}_j)_{j\in\mathbb{N}_0}$ is sub-additive. This justifies the definition of $\overline{s}(\sigma)$. As $\log \underline{\sigma}_j = -\log(\overline{\sigma^{-1}})_j$, $\underline{s}(\sigma)$ also makes sense.
We remark that if σ and β are admissible sequences such that $\sigma \sim \beta$, then their Boyd indices coincide.
In the present paper we will apply frequently the following property: for each $\delta > 0$ there are two positive constants $c_1 = c_1(\delta)$ and $c_2 = c_2(\delta)$ such that for all $j, k \in \mathbb{N}_0$,

$$c_1 2^{(\underline{s}(\sigma)-\delta)j} \le \frac{\sigma_{j+k}}{\sigma_k} \le c_2 2^{(\overline{s}(\sigma)+\delta)j}. \tag{2.2.2}$$

REMARK 2.11. In the context of generalised smoothness Bricchi (cf. [3]) considered

$$\underline{\mathfrak{s}}(\sigma) = \liminf_{j\to\infty} \log\left(\frac{\sigma_{j+1}}{\sigma_j}\right), \quad \overline{\mathfrak{s}}(\sigma) = \limsup_{j\to\infty} \log\left(\frac{\sigma_{j+1}}{\sigma_j}\right).$$

If σ is an admissible sequence, then by (2.2.1) it follows immediately that both $\underline{\mathfrak{s}}(\sigma)$ and $\overline{\mathfrak{s}}(\sigma)$ are well-defined and finite. These indices were used by Bricchi in [3] to deal with Besov spaces with generalised smoothness.
The indices $\underline{s}(\sigma)$ and $\underline{\mathfrak{s}}(\sigma)$ [respectively $\overline{s}(\sigma)$ and $\overline{\mathfrak{s}}(\sigma)$] may not coincide. In the proofs of his results, Bricchi used certain conditions on $\underline{\mathfrak{s}}(\sigma)$ and $\overline{\mathfrak{s}}(\sigma)$ in order to get estimations of the same kind of (2.2.2) with $j = 1$ or $k = 0$. Most conditions in his results can be adapted and written using $\underline{s}(\sigma)$ and $\overline{s}(\sigma)$. We will do that adaptation.

We will also deal with the following particular kind of admissible sequences.

PROPOSITION 2.12. *Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence and Λ an admissible function associated to σ. Let $\alpha > 0$. Then*

$$\sigma_\alpha = (\sigma_{\alpha,j})_{j\in\mathbb{N}_0}, \quad \sigma_{\alpha,j} := \Lambda(2^{\alpha j}) \tag{2.2.3}$$

is an admissible sequence. Furthermore,

$$\underline{s}(\sigma_\alpha) = \alpha \underline{s}(\sigma) \quad \textit{and} \quad \overline{s}(\sigma_\alpha) = \alpha \overline{s}(\sigma). \tag{2.2.4}$$

Let $k \in \mathbb{N}_0$. The sequence

$$T_k(\sigma) = (\sigma_{j+k})_{j\in\mathbb{N}_0}$$

is admissible and

$$\underline{s}(T_k(\sigma)) = \underline{s}(\sigma) \quad \textit{and} \quad \overline{s}(T_k(\sigma)) = \overline{s}(\sigma). \tag{2.2.5}$$

Proof. Step 1. We prove (2.2.4) for $0 < \alpha \le 1$, taking advantage of the fact that, for such α,

$$\{[\alpha k] : k \in \mathbb{N}_0\} = \mathbb{N}_0. \tag{2.2.6}$$

Let $\gamma = \sigma_\alpha$. One can easily see that

$$\frac{\gamma_{j+k}}{\gamma_k} \sim \frac{\sigma_{[\alpha k]+[\alpha j]}}{\sigma_{[\alpha k]}}, \quad \text{for all} \quad j,k \in \mathbb{N}_0. \tag{2.2.7}$$

By (2.2.6) and (2.2.7),

$$\overline{\gamma}_j \sim \overline{\sigma}_{[\alpha j]} \quad \text{and} \quad \underline{\gamma}_j \sim \underline{\sigma}_{[\alpha j]}, \quad j \in \mathbb{N}_0.$$

Hence

$$\underline{s}(\gamma) = \lim_{j\to\infty} \frac{\log \underline{\sigma}_{[\alpha j]}}{j} = \alpha \lim_{j\to\infty} \frac{\log \underline{\sigma}_{[\alpha j]}}{[\alpha j]} \lim_{j\to\infty} \frac{[\alpha j]}{\alpha j} \cdot = \alpha \underline{s}(\sigma).$$

Analogously $\overline{s}(\gamma) = \alpha \overline{s}(\sigma)$.
Step 2. Let $\alpha > 1$ and $\gamma = \sigma_\alpha$. Then $\sigma \sim \gamma_{\alpha^{-1}}$ and so, by Remark 2.10 and Step 1,

$$\underline{s}(\sigma) = \frac{1}{\alpha}\underline{s}(\gamma) \quad \text{and} \quad \overline{s}(\sigma) = \frac{1}{\alpha}\overline{s}(\gamma).$$

Step 3. We prove (2.2.5) for the lower Boyd index. We fix $k \in \mathbb{N}_0$ and consider $\beta = T_k(\sigma)$. For all $j \in \mathbb{N}_0$,

$$\underline{\beta}_j = \inf_{t \in \mathbb{N}_0} \frac{\beta_{j+t}}{\beta_t} = \inf_{t \in \mathbb{N}_0} \frac{\sigma_{j+k+t}}{\sigma_{k+t}} \geq \underline{\sigma}_j.$$

So $\underline{s}(\beta) \geq \underline{s}(\sigma)$.
For all $j \in \mathbb{N}_0$,

$$\underline{\sigma}_j = \inf\left\{ \inf_{t \geq k} \frac{\sigma_{j+t}}{\sigma_t}, \inf_{0 \leq t < k} \frac{\sigma_{j+t}}{\sigma_t} \right\} = \inf\left\{ \underline{\beta}_j, \inf_{0 \leq t < k} \frac{\sigma_{j+t}}{\sigma_t} \right\}.$$

Let $\delta > 0$. Applying (2.2.2),

$$\inf_{0 \leq t < k} \frac{\sigma_{j+t}}{\sigma_t} = \inf_{0 \leq t < k} \left(\frac{\sigma_{j+t}}{\sigma_{j+k+t}} \cdot \frac{\sigma_{j+k+t}}{\sigma_{k+t}} \cdot \frac{\sigma_{k+t}}{\sigma_t} \right) \geq c 2^{-(\overline{s}(\sigma)+\delta)k} 2^{(\underline{s}(\sigma)-\delta)k} \inf_{0 \leq t < k} \frac{\beta_{j+t}}{\beta_t}$$

Hence, for all $j \in \mathbb{N}_0$,

$$\underline{\sigma}_j \geq \min\{1, c 2^{-(\overline{s}(\sigma)+\delta)k} 2^{(\underline{s}(\sigma)-\delta)k}\} \underline{\beta}_j,$$

and then $\underline{s}(\beta) \leq \underline{s}(\sigma)$. ■

Remark 2.13. It follows immediately that, in the conditions of Proposition 2.12, the sequence $T_k(\sigma_\alpha)$ is admissible and its Boyd indices can be expressed by means of the corresponding indices of σ. For $N_\alpha = (2^{j\alpha})_{j \in \mathbb{N}_0}$, the function $\Lambda(2^{\alpha k} \cdot)$ is an admissible function associated to $\{T_k(\sigma_\alpha), N_\alpha\}$.

Remark 2.14. The notation σ_α, denoting a sequence as in (2.2.3), should not be confused with σ_j, denoting a term of the sequence σ. The distinction follows clearly from the context, but we make here the convention that a σ with an index will always denote a sequence as in (2.2.3) whenever the index can potentially assume non-integer values. Nevertheless, in the case when a sequence is named by means of the letter N as in N_α, we reserve the special meaning $(2^{\alpha j})_{j \in \mathbb{N}_0}$ to it, which will be recalled whenever deemed necessary.

3. Besov spaces of generalised smoothness on $\mathbb{R}^n$

3.1. The Fourier-analytical approach.

Definition 3.1. For N as in Convention 2.5 and Remark 2.6 we define the *associated covering* $\Omega^N := (\Omega_j^N)_{j \in \mathbb{N}_0}$ of $\mathbb{R}^n$ by

$$\Omega_j^N = \{\xi \in \mathbb{R}^n : |\xi| \leq N_{j+l_0}\}, \quad j = 0, 1, ... l_0 - 1,$$

and

$$\Omega_j^N = \{\xi \in \mathbb{R}^n : N_{j-l_0} \leq |\xi| \leq N_{j+l_0}\}, \quad j \geq l_0.$$

A system $\varphi^N = (\varphi_j^N)_{j \in \mathbb{N}_0}$ will be called a *(generalised) partition of unity subordinated to* Ω^N if:
(i)

$$\varphi_j^N \in C_0^\infty(\mathbb{R}^n) \text{ and } \varphi_j^N(\xi) \geq 0 \quad \text{if } \xi \in \mathbb{R}^n \text{ for any } j \in \mathbb{N}_0;$$

(ii)

$$\operatorname{supp}\varphi_j^N \subset \Omega_j^N \text{ for any } j \in \mathbb{N}_0;$$

(iii) for any $\alpha \in \mathbb{N}_0^n$ there exists a constant $c_\alpha > 0$ such that for any j $\in \mathbb{N}_0$

$$|D^\alpha \varphi_j^N(\xi)| \le c_\alpha (1+|\xi|^2)^{-\frac{|\alpha|}{2}} \quad \text{for any } \xi \in \mathbb{R}^n;$$

(iv) there exists a constant $c_\varphi > 0$ such that

$$\sum_{j=0}^{\infty} \varphi_j^N(\xi) = c_\varphi < \infty \quad \text{for any } \xi \in \mathbb{R}^n.$$

Let us define Besov spaces of generalised smoothness on $\mathbb{R}^n$, according to [11]:

Definition 3.2. Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence and $0 < p, q \le \infty$. Let $N = (N_j)_{j\in\mathbb{N}_0}$ and $\varphi^N = (\varphi_j^N)_{j\in\mathbb{N}_0}$ be as in Definition 3.1.
The *Besov space of generalised smoothness* on $\mathbb{R}^n$ is given by

$$B_{p,q}^{\sigma,N}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f|B_{p,q}^{\sigma,N}(\mathbb{R}^n)\| = \|(\sigma_j \varphi_j^N(D)f)_{j\in\mathbb{N}_0}|\ell_q(L_p)\| < \infty\}.$$

Remark 3.3. This Fourier analytic description of Besov spaces of generalised smoothness was given in [11]. In this work one can also find some information about the history of function spaces of generalised smoothness with several references.
If $p = q$ we abbreviate $B_p^{\sigma,N}(\mathbb{R}^n) = B_{p,p}^{\sigma,N}(\mathbb{R}^n)$.
If $N_j = 2^j, j \in \mathbb{N}_0$, we have the Besov spaces of generalised smoothness studied by Bricchi in [3] and we write $B_{p,q}^{\sigma}(\mathbb{R}^n)$.

If $N = (2^j)_{j\in\mathbb{N}_0}$ and $\sigma = (s)$ for some $s \in \mathbb{R}$ the above spaces coincide with the usual Besov spaces usually denoted by $B_{p,q}^s(\mathbb{R}^n)$ treated in detail by Triebel in [23], [24] and [26]. We will follow the notation referred above and denote these spaces by $B_{p,q}^{(s)}(\mathbb{R}^n)$.

It is useful for us to deal in the context of $\mathbb{R}^n$ with powers $2^{-\varepsilon j}$, $j \in \mathbb{N}_0$, besides the usual 2^{-j}, $j \in \mathbb{N}_0$. The following proposition clarifies how to switch from one case to the other. It is a consequence of a result proved by Caetano and Leopold (cf. [7], p. 432, Theorem 1). This Theorem is a standardisation result for Besov and Triebel-Lizorkin spaces of generalised smoothness, allowing the reduction, under suitable hypotheses, to corresponding spaces with the usual dyadic decomposition on the Fourier side. Hence, it states that, under some conditions, spaces $B_{p,q}^{\sigma,N}(\mathbb{R}^n)$ can be reduced to spaces $B_{p,q}^{\beta}(\mathbb{R}^n)$ and the construction of the sequence β is given.

Proposition 3.4. *Let* $0 < \varepsilon \le 1$, $N_\varepsilon = (2^{\varepsilon j})_{j\in\mathbb{N}_0}$, σ *be an admissible sequence and* σ_ε *be as in* (2.2.3). *Then*

$$B_{p,q}^{\sigma}(\mathbb{R}^n) = B_{p,q}^{\sigma_\varepsilon,N_\varepsilon}(\mathbb{R}^n)$$

(equivalent quasi-norms).

Remark 3.5. It follows from Proposition 3.4 that, under the same conditions, for all $k \in \mathbb{N}_0$,

$$B_{p,q}^{T_{[\varepsilon k]}(\sigma)}(\mathbb{R}^n) = B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n) \tag{3.1.1}$$

(equivalent quasi-norms), where we are using the notation introduced in Proposition 2.12. Following the proof of [7, Theorem 1, p. 432] one can conclude that the equivalence constants involved in (3.1.1) are independent of k.

3.2. Characterisation by smooth atomic decompositions.

DEFINITION 3.6. Let $0 < \varepsilon \le 1$. We say that

$$\{y^{j,l} : l \in \mathbb{Z}^n\} \subset \mathbb{R}^n,$$

with $j \in \mathbb{N}_0$, is a $2^{-\varepsilon j}$-*approximate lattice* if there exist positive numbers $c_{\varepsilon,1}$ and $c_{\varepsilon,2}$ such that

$$|y^{j,l_1} - y^{j,l_2}| \ge c_{\varepsilon,1} 2^{-\varepsilon j}, j \in \mathbb{N}_0, l_1 \neq l_2, \tag{3.2.1}$$

and

$$\mathbb{R}^n = \bigcup_{l \in \mathbb{Z}^n} B(y^{j,l}, c_{\varepsilon,2} 2^{-\varepsilon j}) \tag{3.2.2}$$

ASSUMPTION 3.7. In what follows, in all results involving approximate lattices, we assume that they are fixed.

EXAMPLE 3.8. Let $0 < \varepsilon \le 1$. For all $j \in \mathbb{N}_0$,

$$\{2^{-\varepsilon j} l : l \in \mathbb{Z}^n\}$$

is a $2^{-\varepsilon j}$-approximate lattice.

When the particular lattices of the above example are considered, they are usually related to the cubes we describe next.

DEFINITION 3.9. Let $0 < \varepsilon \le 1$, $j \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$, $l = (l_1, ..., l_n)$. We denote by $Q_{\varepsilon j,l}$ the half-open cube in $\mathbb{R}^n$ with center at $2^{-\varepsilon j} l$, sides parallel to the coordinate axes and side length $2^{-\varepsilon j}$,

$$Q_{\varepsilon j,l} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : \frac{-2^{-\varepsilon j}}{2} < 2^{-\varepsilon j} l_i - x_i \le \frac{2^{-\varepsilon j}}{2},\ i = 1, ..., n\}.$$

DEFINITION 3.10. Let $0 < \varepsilon \le 1$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and $0 < p \le \infty$. We denote by $\chi^{(p)}_{\varepsilon j,m}$ the p-normalised characteristic function of the cube $Q_{\varepsilon j,m}$, i.e.,

$$\chi^{(p)}_{\varepsilon j,m} = 2^{\frac{\varepsilon j n}{p}} \text{ if } x \in Q_{\varepsilon j,m} \quad \text{and} \quad \chi^{(p)}_{\varepsilon j,m} = 0 \text{ if } x \notin Q_{\varepsilon j,m}.$$

DEFINITION 3.11. Let $0 < p, q \le \infty$, $0 < \varepsilon \le 1$ and

$$\lambda = \{\lambda_{j,l} \in \mathbb{C} : j \in \mathbb{N}_0, l \in \mathbb{Z}^n\}.$$

Then

$$\|\lambda|b_{p,q}\| := \left(\sum_{j=0}^{\infty} \left\| \sum_{l \in \mathbb{Z}^n} \lambda_{j,l} \chi^{(p)}_{\varepsilon j,l} | L_p(\mathbb{R}^n) \right\|^q \right)^{\frac{1}{q}} = \left(\sum_{j=0}^{\infty} \left(\sum_{l \in \mathbb{Z}^n} |\lambda_{j,l}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

(with the usual modification if $p = \infty$ or $q = \infty$) and

$$b_{p,q} := \left\{ \lambda : \|\lambda|b_{p,q}\| < \infty \right\}.$$

If $p = q$ we use the abbreviation $b_p = b_{p,p}$.

Definition 3.12. Let $K \in \mathbb{N}_0$ and $0 < \varepsilon \leq 1$. Consider, for all $j \in \mathbb{N}_0$, a fixed $2^{-\varepsilon j}$-approximate lattice as in Definition 3.6. Let $d > c_{\varepsilon,2}$, where $c_{\varepsilon,2}$ is as in (3.2.2). Consider an admissible sequence σ and Λ an admissible function associated to σ. For $k \in \mathbb{N}_0$, consider σ_ε as in (2.2.3).
(i) A function $a \in \mathcal{C}^K(\mathbb{R}^n)$ is called a *d-σ-1_K-ε-atom* if

$$\operatorname{supp} a \subset B(y^{0,l}, d), \qquad \text{for some} \quad l \in \mathbb{Z}^n,$$

and

$$\sup_{x \in \mathbb{R}^n} |D^\alpha a(x)| \leq \sigma_0^{-1} \quad \text{for} \quad |\alpha| \leq K.$$

(ii) Additionally, let $0 < p \leq \infty$ and $L \in \mathbb{N}_0 \cup \{-1\}$. A function $a \in \mathcal{C}^K(\mathbb{R}^n)$ is called a *d-$(\sigma, p)_{K,L}$-ε-atom* if for some $j \in \mathbb{N}$

$$\operatorname{supp} a \subset B(y^{j,l}, d2^{-\varepsilon j}) \quad \text{for some} \quad l \in \mathbb{Z}^n;$$

$$\sup_{x \in \mathbb{R}^n} |D^\alpha a(x)| \leq \sigma_j^{-1}\, 2^{\frac{\varepsilon j n}{p}}\, 2^{\varepsilon|\alpha| j} \quad \text{for} \quad |\alpha| \leq K.$$

and

$$\int_{\mathbb{R}^n} x^\beta a(x) dx = 0 \quad \text{if} \quad |\beta| \leq L. \tag{3.2.3}$$

Remark 3.13. If $L = -1$ then no moment condition (3.2.3) is required. In this case we omit the subscript "L" and we simply speak of d-$(\sigma, p)_K$-ε-atoms. We say for an atom as above that it is located at $B(y^{j,l}, d2^{-\varepsilon j})$ and we shall denote it by $a^{j,l}$.

Theorem 3.14. *Let $0 < p, q \leq \infty$, $k \in \mathbb{N}_0$, $0 < \varepsilon \leq 1$, $N_\varepsilon = (2^{\varepsilon j})_{j \in \mathbb{N}_0}$ and σ be an admissible sequence such that*

$$\underline{s}(\sigma) > n\Big(\frac{1}{p} - 1\Big)_+$$

Fix $K \in \mathbb{N}_0$ with $K > \overline{s}(\sigma)$ and $d > c_{\varepsilon,2}$, for $c_{\varepsilon,2}$ as in (3.2.2). Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^{T_k(\sigma_\varepsilon), N_\varepsilon}_{p,q}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f(x) = \sum_{j=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \nu_{j,l} a^{j,l}(x), \tag{3.2.4}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$, where $a^{j,l}$ are d-$T_k(\sigma_\varepsilon)$-1_K-ε-atoms ($j = 0$) or d-$(T_k(\sigma_\varepsilon), p)_K$-$\varepsilon$-atoms ($j \in \mathbb{N}$) according to Definition 3.12 and Remark 3.13, and $\nu \in b_{p,q}$. Furthermore,

$$\|f | B^{T_k(\sigma_\varepsilon), N_\varepsilon}_{p,q}(\mathbb{R}^n)\| \sim \inf \|\nu | b_{p,q}\| \tag{3.2.5}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (3.2.4). The equivalence constants in (3.2.5) are independent of k.

Remark 3.15. A characterisation of the spaces $B^{\sigma,N}_{p,q}(\mathbb{R}^n)$ with atomic decompositions was proved in [11]. In this work atoms were defined as being located in cubes as in Definition 3.9 with N_j instead of $2^{\varepsilon j}$.
For the next sections it was convenient to have such a result for spaces $B^{T_k(\sigma_\varepsilon), N_\varepsilon}_{p,q}(\mathbb{R}^n)$ taking atoms located in more general approximate lattices, as in Definition 3.6, and also guaranteeing the independence of k in all constants involved. This can be obtained

following directly the proofs of Theorem 4.4.3, p. 49, in [11] and all the intermediate results and relying also on the characterisations of spaces $B^{\sigma,N}_{p,q}(\mathbb{R}^n)$ with quarkonial decompositions located in these more general approximate lattices obtained by Knopova and Zähle in [17].

3.3. Characterisation by differences. We recall the definition of differences of functions. If f is an arbitrary complex-valued function on $\mathbb{R}^n$, $u \in \mathbb{R}^n$ and $M \in \mathbb{N}$, then

$$(\Delta^M_u f)(x) := \sum_{j=0}^{M} \binom{M}{j} (-1)^{M-j} f(x + ju), \quad x \in \mathbb{R}^n,$$

where $\binom{M}{j}$ are the binomial coefficients. The differences of functions can also be defined iteratively via

$$(\Delta^1_u f)(x) = f(x+u) - f(x) \quad \text{and} \quad (\Delta^{k+1}_u f)(x) = \Delta^1_u(\Delta^k_u f)(x), \quad k \in \mathbb{N}.$$

Furthermore, the *k-th modulus of smoothness* of a function $f \in L_p(\mathbb{R}^n)$, $0 < p \leq \infty$, $k \in \mathbb{N}$, is defined by

$$\omega_k(f,t)_p := \sup_{|u|\leq t} \|\Delta^k_u f | L_p(\mathbb{R}^n)\|, \quad t > 0.$$

In [16] equivalent norms involving differences for some Besov spaces of generalised smoothness were presented. In [21], Moura presented a characterisation by differences of Besov spaces with generalised smoothness. In the next Theorem we present equivalent quasi-norms more convenient for what will be done in the next sections.

THEOREM 3.16. *Let $0 < \varepsilon \leq 1$, $N_\varepsilon = (2^{\varepsilon j})_{j\in\mathbb{N}_0}$, $0 < p, q \leq \infty$, σ be an admissible sequence and Λ be an admissible function associated to σ. Consider*

$$\underline{s}(\sigma) > n\Big(\frac{1}{p} - 1\Big)_+ \quad \text{and} \quad \overline{s}(\sigma) < M \in \mathbb{N}. \tag{3.3.1}$$

For any $b \in (0, +\infty)$, $B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in L_{\overline{p}}(\mathbb{R}^n)$, with $\overline{p} = \max(1, p)$, such that

$$\|f | B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_M := \sigma_{\varepsilon,k}\|f|L_p(\mathbb{R}^n)\| + \Big(\int_{|u|\leq b} \big(\Lambda(2^{\varepsilon k}|u|^{-1})\omega_M(f,|u|)_p\big)^q \frac{du}{|u|^n}\Big)^{\frac{1}{q}} \tag{3.3.2}$$

is finite or, equivalently, such that

$$\|f | B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|_M := \sigma_{\varepsilon,k}\|f|L_p(\mathbb{R}^n)\| + \Big(\int_{|u|\leq b} \big(\Lambda(2^{\varepsilon k}|u|^{-1})\|\Delta^M_u f|L_p(\mathbb{R}^n)\|\big)^q \frac{du}{|u|^n}\Big)^{\frac{1}{q}} \tag{3.3.3}$$

is finite (with the usual modification if $q = \infty$).
Moreover, $\|\cdot|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_M$ and $\|\cdot|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|_M$ are equivalent quasi-norms for $B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$ and the related equivalence constants are independent of k.

Proof. *Step 1.* The function $\Lambda(2^{\varepsilon k}\cdot)$ is an admissible function associated to $\{T_k(\sigma_\varepsilon), N_\varepsilon\}$. Hence, according to [21], in the conditions considered,

$$\|f|L_p(\mathbb{R}^n)\| + \left(\int_{|u|\le b} \left(\Lambda(2^{\varepsilon k}|u|^{-1})\omega_M(f,|u|)_p\right)^q \frac{du}{|u|^n}\right)^{\frac{1}{q}} \tag{3.3.4}$$

is an equivalent quasi-norm for the elements of $B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$. In Moura's work there was no interest in taking sequences $T_k(\sigma_\varepsilon)$ and guaranteing the independence of k. Following the proof in [21] adapted to our purposes, we concluded that one has to modify (3.3.4) by (3.3.2) to get equivalent quasi-norms where the related equivalence constants can be chosen independently of k. Moreover, actually $f \in B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$ if, and only if, $f \in L_{\overline{p}}(\mathbb{R}^n)$ and (3.3.2) is finite.

Step 2. It remains to prove that (3.3.2) can be replaced by (3.3.3). If f belongs to $B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$, then one has by embedding and by Step 1 that $f \in L_{\overline{p}}(\mathbb{R}^n)$ and

$$\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|_M \le \|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_M \lesssim \|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|.$$

We now prove the reverse. We follow Triebel's proof for the classical Besov spaces in [23, Section 2.5.12]. We will consider Λ as in Example 2.8 taking there $N = (2^j)_{j\in\mathbb{N}_0}$. Let $f \in L_{\overline{p}}(\mathbb{R}^n)$ such that $\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|_M < \infty$, for some $b > 0$. We fix $\delta \in (0, \underline{s}(\sigma))$. There is $j_\delta \in \mathbb{N}_0$ such that

$$\frac{\sigma_{j+t}}{\sigma_t} \ge 2^{(\underline{s}(\sigma)-\delta)j}, \quad j \ge j_\delta. \tag{3.3.5}$$

Let $m \in \mathbb{N}$ be such that $\varepsilon m \ge j_\delta + 3$. For all $x \in [2^{\varepsilon j}, 2^{\varepsilon(j+1)})$, $j \in \mathbb{N}_0$,

$$\Lambda(2^{\varepsilon k}x) \le \max_{0\le i\le 2} \sigma_{[\varepsilon(j+k)]+i} \quad \text{and} \quad \Lambda(2^{\varepsilon m}\cdot 2^{\varepsilon k}x) \ge \min_{[\varepsilon m]\le l\le[\varepsilon m]+3} \sigma_{[\varepsilon(j+k)]+l},$$

and so, by (3.3.5), we obtain

$$\begin{aligned} \frac{\Lambda(2^{\varepsilon k}x)}{\Lambda(2^{\varepsilon m}\cdot 2^{\varepsilon k}x)} &\le \max_{\substack{0\le i\le 2,\\ [\varepsilon m]\le l\le[\varepsilon m]+3}} \frac{\sigma_{[\varepsilon(j+k)]+i}}{\sigma_{[\varepsilon(j+k)]+i+(l-i)}} \\ &\le \max_{\substack{0\le i\le 2,\\ [\varepsilon m]\le l\le[\varepsilon m]+3}} 2^{-(l-i)(\underline{s}(\sigma)-\delta)} \\ &\le 2^{-(\underline{s}(\sigma)-\delta)}. \end{aligned} \tag{3.3.6}$$

Hence, considering $q < \infty$,

$$\begin{aligned} I = &\int_{|u|\le b} \left(\Lambda(2^{\varepsilon k}|u|^{-1})\omega_M(f,|u|)_p\right)^q \frac{du}{|u|^n} \\ = &\int_{|u|\le b} \left(\Lambda(2^{\varepsilon k}|u|^{-1}) \sup_{|\rho|\le|u|} \|\Delta^M_\rho f|L_p(\mathbb{R}^n)\|\right)^q \frac{du}{|u|^n} \\ \le &\int_{|u|\le b} \left(\Lambda(2^{\varepsilon k}|u|^{-1}) \sup_{2^{-\varepsilon m}|u|\le|\rho|\le|u|} \|\Delta^M_\rho f|L_p(\mathbb{R}^n)\|\right)^q \frac{du}{|u|^n} \\ &+ \int_{|u|\le b} \left(\Lambda(2^{\varepsilon k}|u|^{-1}) \sup_{|\rho|\le 2^{-\varepsilon m}|u|} \|\Delta^M_\rho f|L_p(\mathbb{R}^n)\|\right)^q \frac{du}{|u|^n} \\ \le &\int_{|u|\le b} \left(\Lambda(2^{\varepsilon k}|u|^{-1}) \sup_{2^{-\varepsilon m}|u|\le|\rho|\le|u|} \|\Delta^M_\rho f|L_p(\mathbb{R}^n)\|\right)^q \frac{du}{|u|^n} + 2^{-q(\underline{s}(\sigma)-\delta)}I, \end{aligned}$$

where we used (3.3.6). So,

$$I \lesssim \int_{|u|\le b} \Big(\Lambda(2^{\varepsilon k}|u|^{-1}) \sup_{2^{-\varepsilon m}|u|\le|\rho|\le|u|} \|\Delta_\rho^M f|L_p(\mathbb{R}^n)\|\Big)^q \frac{du}{|u|^n}.$$

Let $\rho = \rho_0 + \rho_1$. Then

$$e^{i\rho\cdot\xi} - 1 = e^{i\rho_0\cdot\xi}(e^{i\rho_1\cdot\xi} - 1) + e^{i\rho_0\cdot\xi} - 1. \tag{3.3.7}$$

Next we will raise (3.3.7) to the power $2M$, $M \in \mathbb{N}$, apply it to $\mathcal{F}f$ and take the inverse Fourier transform of the result in order to obtain

$$\|\Delta_\rho^{2M} f|L_p(\mathbb{R}^n)\|^q \le c\|\Delta_{\rho_0}^M f|L_p(\mathbb{R}^n)\|^q + c\|\Delta_{\rho_1}^M f|L_p(\mathbb{R}^n)\|^q. \tag{3.3.8}$$

We present the calculations to obtain (3.3.8) from (3.3.7):
For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $u \in \mathbb{R}^n$

$$\Big(\mathcal{F}\big(e^{-iu\cdot\xi}(\mathcal{F}^{-1}\varphi)(\xi)\big)\Big)(z) = \big(\mathcal{F}(\mathcal{F}^{-1}\varphi)\big)(z+u) = \varphi(z+u).$$

Hence,

$$\begin{aligned} \Big\langle\Big(\mathcal{F}^{-1}\big(e^{-iu\cdot\xi}(\mathcal{F}f)(\xi)\big)\Big)(y), \varphi(y)\Big\rangle &= \Big\langle f(z), \Big(\mathcal{F}\big(e^{-iu\cdot\xi}(\mathcal{F}^{-1}\varphi)(\xi)\big)\Big)(z)\Big\rangle \\ &= \big\langle f(z), \varphi(z+u)\big\rangle \\ &= \int_{\mathbb{R}^n} f(y-u)\varphi(y)dy. \end{aligned} \tag{3.3.9}$$

So, $\mathcal{F}^{-1}\big(e^{-iu\cdot\xi}(\mathcal{F}f)(\xi)\big)$ is a regular distribution given by $f(y-u)$, $y \in \mathbb{R}^n$.
Now, on the one hand,

$$\big[\mathcal{F}^{-1}\big((e^{i\rho\cdot\xi} - 1)^{2M}(\mathcal{F}f)(\xi)\big)\big](y) = (\Delta_\rho^{2M} f)(y). \tag{3.3.10}$$

On the other hand, by (3.3.7), and writing temporarily $\|\cdot\|_p$ instead of the usual $\|\cdot|L_p(\mathbb{R}^n)\|$,

$$\begin{aligned}
&\|\mathcal{F}^{-1}\big((e^{i\rho\cdot\xi}-1)^{2M}(\mathcal{F}f)(\xi)\big)\|_p \\
&= \Big\|\mathcal{F}^{-1}\Big(\sum_{t=0}^{2M}\tbinom{2M}{t}e^{it\rho_0\cdot\xi}(e^{i\rho_1\cdot\xi}-1)^t(e^{i\rho_0\cdot\xi}-1)^{2M-t}(\mathcal{F}f)(\xi)\Big)\Big\|_p \\
&\le c\sum_{t=0}^{M}\big\|\mathcal{F}^{-1}\big\{e^{it\rho_0\cdot y}\big[\mathcal{F}(\mathcal{F}^{-1}(e^{i\rho_1\cdot\xi}-1)^t(e^{i\rho_0\cdot\xi}-1)^{2M-t}(\mathcal{F}f)(\xi))\big](y)\big\}\big\|_p \\
&\quad + c\sum_{t=M+1}^{2M}\big\|\mathcal{F}^{-1}\big\{e^{it\rho_0\cdot y}\big[\mathcal{F}(\mathcal{F}^{-1}(e^{i\rho_1\cdot\xi}-1)^t(e^{i\rho_0\cdot\xi}-1)^{2M-t}(\mathcal{F}f)(\xi))\big](y)\big\}\big\|_p \\
&\le c\sum_{t=0}^{M}\big\|\mathcal{F}^{-1}\big\{(e^{i\rho_1\cdot y}-1)^t\big[\mathcal{F}\big(\mathcal{F}^{-1}(e^{i\rho_0\cdot\xi}-1)^{2M-t}(\mathcal{F}f)(\xi)\big)\big](y)\big\}\big\|_p \\
&\quad + c\sum_{t=M+1}^{2M}\big\|\mathcal{F}^{-1}\big\{(e^{i\rho_0\cdot y}-1)^{2M-t}\big[\mathcal{F}\big(\mathcal{F}^{-1}(e^{i\rho_1\cdot\xi}-1)^t(\mathcal{F}f)(\xi)\big)\big](y)\big\}\big\|_p \\
&= c\sum_{t=0}^{M}\big\|\Delta_{\rho_1}^t\big[\mathcal{F}^{-1}(e^{i\rho_0\cdot\xi}-1)^{2M-t}(\mathcal{F}f)(\xi)\big]\big\|_p \\
&\quad + c\sum_{t=M+1}^{2M}\big\|\Delta_{\rho_0}^{2M-t}\big[\mathcal{F}^{-1}(e^{i\rho_1\cdot\xi}-1)^t(\mathcal{F}f)(\xi)\big]\big\|_p \\
&\le c'\sum_{t=0}^{M}\big\|\mathcal{F}^{-1}(e^{i\rho_0\cdot\xi}-1)^{2M-t}(\mathcal{F}f)(\xi)\big\|_p + c'\sum_{t=M+1}^{2M}\big\|\mathcal{F}^{-1}(e^{i\rho_1\cdot\xi}-1)^t(\mathcal{F}f)(\xi)\big\|_p \\
&= c'\sum_{t=0}^{M}\|\Delta_{\rho_0}^{2M-t}f\|_p + c'\sum_{t=M+1}^{2M}\|\Delta_{\rho_1}^t f\|_p \\
&\le c''\|\Delta_{\rho_0}^M f\|_p + c''\|\Delta_{\rho_1}^M f\|_p,
\end{aligned}$$

where we also applied (3.3.9) and (3.3.10), and the constants involved are independent of ρ, ρ_0, ρ_1 and f. So we proved (3.3.8).

Let $\rho \in \mathbb{R}^n$ with $2^{-\varepsilon m}|u| \le |\rho| \le |u|$ and consider K the ball represented in Figure 1.

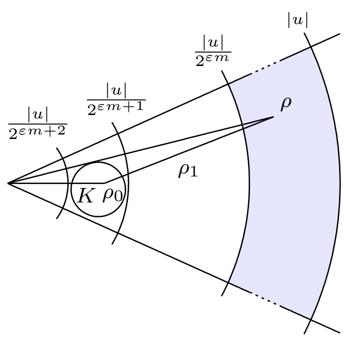

Fig. 1.

Integrating (3.3.8) over K, we obtain

$$\begin{aligned}\|\Delta_\rho^{2M} f|L_p(\mathbb{R}^n)\|^q &\le c \int\limits_{\frac{|u|}{2^{\varepsilon m+2}}\le|\lambda|\le|u|} \|\Delta_\lambda^M f|L_p(\mathbb{R}^n)\|^q d\lambda\cdot|u|^{-n}\\ &\le c' \int\limits_{2^{-\varepsilon m-2}\le|\lambda|\le 1} \|\Delta_{\lambda|u|}^M f|L_p(\mathbb{R}^n)\|^q d\lambda\\ &\le c\int_{2^{-\varepsilon m-2}}^1\int_{\omega_n} \|\Delta_{rv\omega}^M f|L_p(\mathbb{R}^n)\|^q d\omega dv,\end{aligned}$$

with $r=|u|$ and polar coordinates v and $\omega\in\omega_n$ where the latter stands for the unit sphere in $\mathbb{R}^n$.
We take the supremum with respect to ρ, where $\frac{|u|}{2^{\varepsilon m}}\le|\rho|\le|u|$, we multiply by $\Lambda(2^{\varepsilon k}|u|^{-1})^q\cdot|u|^{-n}$, and integrate, obtaining

$$\begin{aligned}&\int\limits_{|u|\le b}\Lambda(2^{\varepsilon k}|u|^{-1})^q \sup_{\frac{|u|}{2^{\varepsilon m}}\le|\rho|\le\|u|} \|\Delta_\rho^{2M} f|L_p(\mathbb{R}^n)\|^q\frac{du}{|u|^n}\\ &=\int_0^b\Lambda(2^{\varepsilon k}r^{-1})^q\sup_{\frac{r}{2^{\varepsilon m}}\le|\rho|\le r}\|\Delta_\rho^{2M} f|L_p(\mathbb{R}^n)\|^q\frac{dr}{r}\\ &\le c\int_{2^{-\varepsilon m-2}}^1\int_{\omega_n}\int_0^b\Lambda(2^{\varepsilon k}r^{-1})^q\|\Delta_{rv\omega}^M f|L_p(\mathbb{R}^n)\|^q\frac{dr}{r}d\omega dv\\ &\le c'\int\limits_{|u|\le b}\Lambda(2^{\varepsilon k}|u|^{-1})^q\|\Delta_u^M f|L_p(\mathbb{R}^n)\|^q\frac{du}{|u|^n},\end{aligned}$$

where we also applied (2.2.2) and the constants are independent of k. Hence, for $0<q<\infty$, we proved that if $f\in L_{\overline{p}}(\mathbb{R}^n)$ with $\|f|B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|_M<\infty$, for some $b>0$, then

$$\|f|B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|_{2M}^*\lesssim\|f|B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|_M.$$

The case $q=\infty$ is proved making the usual modifications. By Step 1, we conclude. ∎

In the next Proposition we present equivalent quasi-norms for the elements of certain subspaces of Besov spaces of generalised smoothness on $\mathbb{R}^n$. It will be an important tool to prove the adapted homogeneity property for spaces of generalised smoothness.

Proposition 3.17. *Let Ω be a bounded domain in $\mathbb{R}^n$. Consider an admissible sequence σ and Λ an admissible function associated to σ.*

$$0<p,q\le\infty,\quad \underline{s}(\sigma)>n\Big(\frac{1}{p}-1\Big)_+\quad\text{and}\quad\overline{s}(\sigma)<M\in\mathbb{N}.$$

Let $0<\varepsilon\le 1$, $N_\varepsilon=(2^{\varepsilon j})_{j\in\mathbb{N}_0}$, $b>0$ and $k\in\mathbb{N}_0$. Then

$$\|f|B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|\sim\bigg(\int_{|u|\le b}\big(\Lambda(2^{\varepsilon k}|u|^{-1})\|\Delta_u^M f|L_p(\mathbb{R}^n)\|\big)^q\frac{du}{|u|^n}\bigg)^{\frac{1}{q}},\tag{3.3.11}$$

(with the usual modification if $q=\infty$), for all

$$f\in B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\quad\text{with}\quad\operatorname{supp} f\subset\overline{\Omega}.\tag{3.3.12}$$

The equivalence constants in (3.3.11) *are independent of f and k.*

Proof. We present the proof for $0 < q < \infty$. The case $q = \infty$ is proved making the usual adaptations.
One of the inequalities follows immediately from Theorem 3.16. So we just have to prove that under the above conditions there is a positive number c such that

$$\|f|L_p(\mathbb{R}^n)\| \leq c\sigma_{\varepsilon,k}^{-1}\left(\int_{|u|\leq b}\left(\Lambda(2^{\varepsilon k}|u|^{-1})\|\Delta_u^M f|L_p(\mathbb{R}^n)\|\right)^q \frac{du}{|u|^n}\right)^{\frac{1}{q}},$$

for all $k \in \mathbb{N}_0$ and for all f as in (3.3.12).
As $\operatorname{supp} f$ is contained in a fixed bounded set, for $0 < p < 1$, we have

$$\|f|L_p(\mathbb{R}^n)\| \leq c\|f|L_1(\mathbb{R}^n)\|.$$

Hence it is enough to prove that, for $0 < p \leq \infty$,

$$\|f|L_{\overline{p}}(\mathbb{R}^n)\| \leq c\sigma_{\varepsilon,k}^{-1}\left(\int_{|u|\leq b}\left(\Lambda(2^{\varepsilon k}|u|^{-1})\|\Delta_u^M f|L_p(\mathbb{R}^n)\|\right)^q \frac{du}{|u|^n}\right)^{\frac{1}{q}},$$

for all $k \in \mathbb{N}_0$ and for all f as in (3.3.12).
We can suppose that $b \leq 1$. Therefore, considering $\delta \in (0, \underline{s}(\sigma) - n(\frac{1}{p} - 1)_+)$ and choosing $J \in \mathbb{N}_0$ such that $2^{-\varepsilon J} \leq b$, we obtain

$$\begin{aligned}
&\sigma_{\varepsilon,k}^{-q}\int_{|u|\leq b}\left(\Lambda(2^{\varepsilon k}|u|^{-1})\|\Delta_u^M f|L_p(\mathbb{R}^n)\|\right)^q \frac{du}{|u|^n}\\
&\quad\geq \sum_{j=J}^{\infty}\int_{2^{-\varepsilon(j+1)}\leq|u|\leq 2^{-\varepsilon j}}\left(\frac{\Lambda(2^{\varepsilon k}|u|^{-1})}{\sigma_{\varepsilon,k}}\|\Delta_u^M f|L_p(\mathbb{R}^n)\|\right)^q \frac{du}{|u|^n}\\
&\quad\geq c\int_{|u|\leq 2^{-\varepsilon J}}|u|^{-(\underline{s}(\sigma)-\delta)q}\|\Delta_u^M f|L_p(\mathbb{R}^n)\|^q \frac{du}{|u|^n},
\end{aligned} \tag{3.3.13}$$

where we used that fact that, for $2^{-\varepsilon(j+1)} \leq |u| \leq 2^{-\varepsilon j}$,

$$\frac{\Lambda(2^{\varepsilon k}|u|^{-1})}{\sigma_{\varepsilon,k}} \sim \frac{\sigma_{\varepsilon,j+k}}{\sigma_{\varepsilon,k}} \geq c_1 2^{(\underline{s}(\sigma_\varepsilon)-\varepsilon\delta)j} \sim |u|^{-(\underline{s}(\sigma)-\delta)}.$$

By Theorem 3.16, applying (2.2.2) and (3.3.13) we can easily prove that

$$B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n) \subset B_{p,q}^{(\underline{s}(\sigma)-\delta)}(\mathbb{R}^n), \quad \text{for all} \quad k \in \mathbb{N}_0, \tag{3.3.14}$$

where the constants implicitly involved are independent of k.

By (3.3.13) and (3.3.14) we just have to prove that there is a number c such that, for all $f \in B_{p,q}^{(\underline{s}(\sigma)-\delta)}(\mathbb{R}^n)$ with $\operatorname{supp} f \subset \overline{\Omega}$,

$$\|f|L_{\overline{p}}(\mathbb{R}^n)\| \leq c\left(\int_{|u|\leq 2^{-\varepsilon J}}|u|^{-(\underline{s}(\sigma)-\delta)q}\|\Delta_u^M f|L_p(\mathbb{R}^n)\|^q \frac{du}{|u|^n}\right)^{\frac{1}{q}}. \tag{3.3.15}$$

Now one is in the same situation as in [8]. In that case the integration region was $\mathbb{R}^n$ instead of $\{u : |u| \leq 2^{-\varepsilon J}\}$ but this is immaterial. Following the same arguments we can prove that if there is no c such that (3.3.15) is true for all admitted f, then there is

$g \in B^{(\underline{s}(\sigma)-\delta)}_{p,q}(\mathbb{R}^n)$ with $\operatorname{supp} g \subset \overline{\Omega}$, such that

$$\int_{|u|\le 2^{-\varepsilon J}} |u|^{-(\underline{s}(\sigma)-\delta)q} \|\Delta^M_u g|L_p(\mathbb{R}^n)\|^q \frac{du}{|u|^n} = 0 \tag{3.3.16}$$

and

$$\|g|_{B_R}|L_{\overline{p}}(B_R)\| = 1, \tag{3.3.17}$$

where $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ is a fixed ball such that $\overline{\Omega} \subset B_R$. From (3.3.16) follows that

$$(\Delta^M_u g)(x) = 0 \quad \text{for almost all} \quad |u| \le 2^{-\varepsilon J} \quad \text{and} \quad x \in \mathbb{R}^n. \tag{3.3.18}$$

Hence

$$\int_{|u|\le 2^{-\varepsilon J}} |u|^{-2s} \|\Delta^M_u g|L_v(\mathbb{R}^n)\|^2 \frac{du}{|u|^n} = 0, \quad 0 < v \le \infty, \quad s \in \mathbb{R}. \tag{3.3.19}$$

If $p \ge 2$ then $L_p(B_R) \subset L_2(B_R)$ and so, by the conditions on the support of g, $g \in L_2(\mathbb{R}^n)$. Hence, by (3.3.19) and Theorem 3.16, g (identified with its restriction to B_R) belongs to $B^{(m)}_{2,2}(B_R)$, for all $m \in \mathbb{N}$. If $0 < p < 2$ then by (3.3.19) and Theorem 3.16, g (identified with its restriction to B_R) belongs to $B^{(s)}_{p,2}(B_R)$, for all $s \in \mathbb{R}$, in particular, for all $s = m + n(\frac{1}{p} - \frac{1}{2})$, $m \in \mathbb{N}$. Therefore, by a well-known embedding (cf. [23, p. 196, Theorem 3.3.1], for example), $g \in B^{(m)}_{2,2}(B_R)$, for all $m \in \mathbb{N}$. So, for all $0 < p \le \infty$, g (identified with its restriction to B_R) belongs to $B^{(m)}_{2,2}(B_R)$, for all $m \in \mathbb{N}$. As these spaces coincide with the Sobolev spaces $W^m_2(B_R)$, (cf. [23, p. 88, 2.5.6]), then $g \in C^\infty(\overline{B_R})$. This follows from [10, p. 241, Theorem 3.20] (we also refer to pages 202 and 222 for notation). According to [29, p. 201, Remark 4.11], as $g \in C^\infty(\overline{B_R})$ and satisfies (3.3.18), then g must be (locally) polynomial of degree less than M. By compactness arguments it follows that g is globally in B_R a polynomial of degree less than M. As $\operatorname{supp} g \in \overline{\Omega} \subset B_R$, then $g = 0$. But this contradicts (3.3.17). ■

Next, we present and prove an adapted homogeneity property for Besov spaces of generalised smoothness.

THEOREM 3.18. *Let $0 < p, q \le \infty$ and σ be an admissible sequence such that*

$$\underline{s}(\sigma) > n\Big(\frac{1}{p} - 1\Big)_+.$$

Fix $c_0 > 0$ and $0 < \varepsilon \le 1$. Let $N_\varepsilon = (2^{\varepsilon j})_{j\in\mathbb{N}_0}$. Then

$$\|f|B^{\sigma_\varepsilon,N_\varepsilon}_{p,q}(\mathbb{R}^n)\| \sim 2^{-\frac{\varepsilon k n}{p}} \|f(2^{-\varepsilon k}\cdot)|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\| \tag{3.3.20}$$

for all $k \in \mathbb{N}_0$ and all

$$f \in B^{\sigma_\varepsilon,N_\varepsilon}_{p,q}(\mathbb{R}^n) \quad \textit{with} \quad \operatorname{supp} f \subset \{x \in \mathbb{R}^n : |x - x_0| \le c_0 2^{-\varepsilon k}\},$$

for some $x_0 \in \mathbb{R}^n$. The equivalence constants in (3.3.20) are independent of x_0, k and f.

Proof. Step 1. Let $x_0 \in \mathbb{R}^n$ be such that $\operatorname{supp} f \subset \{x \in \mathbb{R}^n : |x - x_0| \le c_0 2^{-\varepsilon k}\}$. We may assume that $x_0 = 0$, otherwise we consider first $f(\cdot + x_0)$ and $f(2^{-\varepsilon k}\cdot + x_0)$ and then conclude for f and $f(2^{-\varepsilon k}\cdot)$. Let Λ be an admissible function associated to σ. Let $f \in B^{\sigma_\varepsilon,N_\varepsilon}_{p,q}(\mathbb{R}^n)$ be such that $\operatorname{supp} f \subset \{x \in \mathbb{R}^n : |x| \le c_0 2^{-\varepsilon k}\}$. We remark that both $\operatorname{supp} f$ and $\operatorname{supp} f(2^{-\varepsilon k}\cdot)$ are contained in $\{x \in \mathbb{R}^n : |x| < c_0\}$. Clearly,

$f(2^{-\varepsilon k}\cdot) \in L_{\overline{p}}(\mathbb{R}^n)$ and $\sigma_{\varepsilon,k}\|f(2^{-\varepsilon k}\cdot)|L_p(\mathbb{R}^n)\|$ is finite. For a fixed $b > 0$ and an integer M such that $M > \overline{s}(\sigma)$,

$$\begin{aligned}
&\left(\int_{|u|\leq b} \left(\Lambda(2^{\varepsilon k}|u|^{-1})\|\Delta_u^M(f(2^{-\varepsilon k}\cdot))|L_p(\mathbb{R}^n)\|\right)^q \frac{du}{|u|^n}\right)^{\frac{1}{q}} \\
&\quad = 2^{\frac{\varepsilon k n}{p}}\left(\int_{|u|\leq b} \left(\Lambda(2^{\varepsilon k}|u|^{-1})\|\Delta_{2^{-\varepsilon k}u}^M f|L_p(\mathbb{R}^n)\|\right)^q \frac{du}{|u|^n}\right)^{\frac{1}{q}} \\
&\quad = 2^{\frac{\varepsilon k n}{p}}\left(\int_{|u|\leq 2^{-\varepsilon k}b} \left(\Lambda(|u|^{-1})\|\Delta_u^M f|L_p(\mathbb{R}^n)\|\right)^q \frac{du}{|u|^n}\right)^{\frac{1}{q}} \\
&\quad \lesssim 2^{\frac{\varepsilon k n}{p}}\|f|B_{p,q}^{\sigma_\varepsilon,N_\varepsilon}(\mathbb{R}^n)\|, \qquad (3.3.21)
\end{aligned}$$

where we applied Theorem 3.16 in the last estimate. Again by Theorem 3.16 we conclude that $f(2^{-\varepsilon k}\cdot) \in B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)$. Now

$$\|f(2^{-\varepsilon k}\cdot)|B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\| \lesssim 2^{\frac{\varepsilon k n}{p}}\|f|B_{p,q}^{\sigma_\varepsilon,N_\varepsilon}(\mathbb{R}^n)\|.$$

follows from Proposition 3.17 and (3.3.21).

Step 2. As $f(2^{-\varepsilon k}\cdot) \in B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)$ then, by Theorem 3.14, there is $\nu \in b_{p,q}$ such that

$$f(2^{-\varepsilon k}x) = \sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n} \nu_{j,l}a^{j,l}(x),$$

where $a^{j,l}$ are, respectively, d-$T_k(\sigma_\varepsilon)$-1_K-ε-atoms and d-$(T_k(\sigma_\varepsilon),p)_K$-$\varepsilon$-atoms related to the $2^{-\varepsilon j}$-approximate lattices, $j \in \mathbb{N}_0$, described in Example 3.8, for some conveniently chosen and fixed K and d. Moreover, ν can be chosen such that

$$\|\nu|b_{p,q}\| \leq c\|f(2^{-\varepsilon k}\cdot)|B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|, \qquad (3.3.22)$$

where c is a positive number independent of f and k. We can easily check that the functions defined by

$$b^{j+k,l}(x) := 2^{\frac{\varepsilon k n}{p}}a^{j,l}(2^{\varepsilon k}x)$$

are d-$(\sigma_\varepsilon,p)_K$-ε-atoms. Then,

$$f(x) = \sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n} \eta_{j+k,l}b^{j+k,l}(x), \quad \text{with} \quad \eta_{j+k,l} := 2^{-\frac{\varepsilon k n}{p}}\nu_{j,l},$$

and, therefore,

$$\|f|B_{p,q}^{\sigma_\varepsilon,N_\varepsilon}(\mathbb{R}^n)\| \lesssim \|\eta|b_{p,q}\| = 2^{-\frac{\varepsilon k n}{p}}\|\nu|b_{p,q}\|$$

Hence, by (3.3.22), we obtain

$$\|f|B_{p,q}^{\sigma_\varepsilon,N_\varepsilon}(\mathbb{R}^n)\| \leq 2^{-\frac{\varepsilon k n}{p}}c\|f(2^{-\varepsilon k}\cdot)|B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|.$$

This concludes the proof. ∎

Remark 3.19. It follows immediately from the above proof that $f \in B_{p,q}^{\sigma_\varepsilon,N_\varepsilon}(\mathbb{R}^n)$ if, and only if, $f(2^{-\varepsilon k}\cdot) \in B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)$.

Remark 3.20. Let us compare (3.3.20) with the homogeneity property obtained for the Besov spaces of classical smoothness $B^{(s)}_{p,q}(\mathbb{R}^n)$ according to Remark 3.3. So, we consider

$$0 < p, q \leq \infty, \quad \varepsilon = 1 \quad \text{and} \quad \sigma = (s) \quad \text{with} \quad s > n\Big(\frac{1}{p} - 1\Big)_+ .$$

According to (3.3.20), for all

$$f \in B^{(s)}_{p,q}(\mathbb{R}^n) \quad \text{with} \quad \operatorname{supp} f \subset \{x \in \mathbb{R}^n : |x - x_0| \leq c_0 2^{-k}\}, \tag{3.3.23}$$

for some $x_0 \in \mathbb{R}^n$, we have

$$\|f|B^{(s)}_{p,q}(\mathbb{R}^n)\| \sim 2^{-\frac{kn}{p}} \|f(2^{-k}\cdot)|B^{T_k((s))}_{p,q}(\mathbb{R}^n)\|. \tag{3.3.24}$$

For all $g \in B^{(s)}_{p,q}(\mathbb{R}^n)$,

$$\begin{aligned} \|g|B^{T_k((s))}_{p,q}(\mathbb{R}^n)\| &\sim \big\|\big(2^{(j+k)s}\varphi_j(D)g\big)_{j\in\mathbb{N}_0}|\ell_q(L_p)\big\| \\ &= 2^{ks}\big\|\big(2^{js}\varphi_j(D)g\big)_{j\in\mathbb{N}_0}|\ell_q(L_p)\big\| \\ &\sim 2^{ks}\|g|B^{(s)}_{p,q}(\mathbb{R}^n)\|, \end{aligned}$$

where $\varphi = (\varphi_j)_{j\in\mathbb{N}_0}$ is some partition of unity according to Definition 3.1. Hence we can rewrite (3.3.24). For all f as in (3.3.23),

$$\|f|B^{(s)}_{p,q}(\mathbb{R}^n)\| \sim 2^{k(s-\frac{n}{p})}\|f(2^{-k}\cdot)|B^{(s)}_{p,q}(\mathbb{R}^n)\|,$$

which corresponds to the homogeneity property for Besov spaces (cf. [8]).

3.4. Characterisation by non-smooth atomic decompositions. In this subsection we present decompositions with non-smooth atoms for the elements of certain Besov spaces of generalised smoothness on $\mathbb{R}^n$. Our interest in non-smooth atoms is connected with function spaces on h-spaces, as it was mentioned in the Introduction. The definition we present next is a generalisation of the one introduced by Triebel in [27]. We use the abbreviations according to Remark 3.3. In particular $B^{(a)\sigma}_p(\mathbb{R}^n) = B^{(a)\sigma}_{p,p}(\mathbb{R}^n)$ has the meaning as explained there.

Definition 3.21. Let $0 < \varepsilon \leq 1$ and $0 < p \leq \infty$. For $j \in \mathbb{N}_0$, let $\{y^{j,l}\}_{l\in\mathbb{Z}^n}$ be a $2^{-\varepsilon j}$-approximate lattice as in Definition 3.6. Let $d > c_{\varepsilon,2}$, where $c_{\varepsilon,2}$ is as in (3.2.2). We fix $a > 0$, consider an admissible sequence σ and

$$(a)\sigma := (2^{aj}\sigma_j)_{j\in\mathbb{N}_0}$$

Then $a^{j,l} \in B^{(a)\sigma}_p(\mathbb{R}^n)$ is called a *d-$(\sigma, p)_a$-ε-atom* if

$$\operatorname{supp} a^{j,l} \subset B(y^{j,l}, d2^{-\varepsilon j}), \qquad j \in \mathbb{N}_0, \quad l \in \mathbb{Z}^n,$$

and

$$\|a^{j,l}|B^{(a)\sigma}_p(\mathbb{R}^n)\| \leq 2^{\varepsilon a j} \tag{3.4.1}$$

The proofs of the next Proposition and Theorem follow the proofs for the classical case in [27]. The most important step was to find the “substitute” for the homogeneity property, which was presented in the above subsection.

In the next Proposition we assert that non-smooth atoms are correctly normalised.

Proposition 3.22. *Let $0 < \varepsilon \le 1$, $0 < p \le \infty$, $d > c_{\varepsilon,2}$ and σ be an admissible sequence such that*

$$\underline{s}(\sigma) > n\Big(\frac{1}{p} - 1\Big)_+.$$

We fix $a > 0$ and consider $a^{j,l}$, with $j \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$, a d-$(\sigma,p)_a$-ε-atom according to Definition 3.21. Then

$$\|a^{j,l}|B_p^\sigma(\mathbb{R}^n)\| \lesssim 1 \qquad \text{and} \qquad \|a^{j,l}|L_p(\mathbb{R}^n)\| \lesssim \sigma_{\varepsilon,j}^{-1}.$$

Proof. Let $N_\varepsilon = (2^{\varepsilon j})_{j\in\mathbb{N}_0}$, β denote the sequence $(a)\sigma$ and Λ be an admissible function associated to σ. Then $\Lambda(2^{\varepsilon j}\cdot)(2^{\varepsilon j}\cdot)^a$ is an admissible function associated to $\{T_j(\beta_\varepsilon), N_\varepsilon\}$.

For all $j \in \mathbb{N}_0$, $l \in \mathbb{Z}^n$, $a^{j,l} \in B_p^\beta(\mathbb{R}^n)$. Then, by Proposition 3.4, $a^{j,l} \in B_p^{\beta_\varepsilon,N_\varepsilon}(\mathbb{R}^n)$ and, by Remark 3.19, $a^{j,l}(2^{-\varepsilon j}\cdot) \in B_p^{T_j(\beta_\varepsilon),N_\varepsilon}(\mathbb{R}^n)$. Hence $a^{j,l}(2^{-\varepsilon j}\cdot + y^{j,l})$ belongs to $B_p^{T_j(\beta_\varepsilon),N_\varepsilon}(\mathbb{R}^n)$ and $\operatorname{supp} a^{j,l}(2^{-\varepsilon j}\cdot + y^{j,l}) \subset \{x : |x| \le d\}$, for all $j \in \mathbb{N}_0$, $l \in \mathbb{Z}^n$. As $a > 0$, the last assertion remains true if we replace β by σ. So, applying Proposition 3.17, we get for some $M > a + \overline{s}(\sigma)$,

$$\begin{aligned}
\|a^{j,l}(2^{-\varepsilon j}\cdot)|B_p^{T_j(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|^p &\sim \int_{|u|\le 1} \Lambda(2^{\varepsilon j}|u|^{-1})^p \|\Delta_u^M f|L_p(\mathbb{R}^n)\|^p \frac{du}{|u|^n} \\
&\le 2^{-\varepsilon j a p} \int_{|u|\le 1} \Lambda(2^{\varepsilon j}|u|^{-1})^p (2^{\varepsilon j a}|u|^{-a})^p \|\Delta_u^M f|L_p(\mathbb{R}^n)\|^p \frac{du}{|u|^n} \\
&\sim 2^{-\varepsilon j a p}\|a^{j,l}(2^{-\varepsilon j}\cdot)|B_p^{T_j(\beta_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|^p,
\end{aligned}$$

where the equivalence constants are independent of j. So, by Proposition 3.4, Theorem 3.18 and also by (3.4.1),

$$\begin{aligned}
\|a^{j,l}|B_p^\sigma(\mathbb{R}^n)\| &\sim \|a^{j,l}|B_p^{\sigma_\varepsilon,N_\varepsilon}(\mathbb{R}^n)\| \\
&\lesssim 2^{-\frac{\varepsilon j n}{p}} 2^{-\varepsilon j a}\|a^{j,l}(2^{-\varepsilon j}\cdot)|B_p^{T_j(\beta_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\| \\
&\sim 2^{-\varepsilon j a}\|a^{j,l}|B_p^{\beta_\varepsilon,N_\varepsilon}(\mathbb{R}^n)\| \\
&\sim 2^{-\varepsilon j a}\|a^{j,l}|B_p^{\beta}(\mathbb{R}^n)\| \\
&\lesssim 1.
\end{aligned}$$

By Theorem 3.16, there is a positive number c, independent of j, such that

$$\|f|L_p(\mathbb{R}^n)\| \le c2^{-\varepsilon j a}\sigma_{\varepsilon,j}^{-1}\|f|B_p^{T_j(\beta_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|, \quad \text{for all} \quad f \in B_p^{T_j(\beta_\varepsilon),N_\varepsilon}(\mathbb{R}^n).$$

Hence

$$\|a^{j,l}|L_p(\mathbb{R}^n)\| \lesssim 2^{-\frac{\varepsilon j n}{p}} 2^{-\varepsilon j a}\sigma_{\varepsilon,j}^{-1}\|a^{j,l}(2^{-\varepsilon j}\cdot)|B_p^{T_j(\beta_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|$$

and again by Theorem 3.18, Proposition 3.4 and (3.4.1) we conclude. ∎

Let b_p be as in Definition 3.11.

Theorem 3.23. *Let $0 < \varepsilon \le 1$, $0 < p \le \infty$ and σ be an admissible sequence such that*

$$\underline{s}(\sigma) > n\Big(\frac{1}{p} - 1\Big)_+.$$

Let $d > c_{\varepsilon,2}$ and $a > 0$. Then $B_p^\sigma(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ which can be *represented as*

$$f(x) = \sum_{j=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \nu_{j,l} a^{j,l}(x), \tag{3.4.2}$$

where $\nu = (\nu_{j,l})_{j,l} \in b_p$ *and* $a^{j,l}$ *are* d*-*$(\sigma, p)_a$*-*ε*-atoms. Furthermore,*

$$\|f|B_p^\sigma(\mathbb{R}^n)\| \sim \inf \|\nu|b_p\|$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (3.4.2).

Proof. Step 1. Let $K > a + \overline{s}(\sigma)$. We have Theorem 3.14, which guarantees that all the elements of $B^\sigma_{p,q}(\mathbb{R}^n)$ admit representations with d-$(\sigma_\varepsilon, p)_K$-$\varepsilon$-atoms according to Definition 3.12. We consider such an atom $a^{j,l}$. We can easily check that $2^{-\varepsilon j a} a^{j,l}$ is an d-$(((a)\sigma)_\varepsilon, p)_K$-$\varepsilon$-atom. Hence, by Theorem 3.14, we get to

$$\|a^{j,l}|B_p^{(a)\sigma}(\mathbb{R}^n)\| \lesssim 2^{\varepsilon a j}$$

and so, as the condition for $\operatorname{supp} a^{j,l}$ is also satisfied, we conclude that, up to constants, $a^{j,l}$ is a d-$(\sigma, p)_a$-ε-atom.

Step 2. It remains to prove that there is a number $c > 0$ such that for all f as in (3.4.2),

$$\|f|B_p^\sigma(\mathbb{R}^n)\| \le c\|\nu|b_p\|.$$

We will prove it for $0 < p < \infty$. The case $p = \infty$ is proved similarly with the usual adaptations. Let $N_\varepsilon = (2^{\varepsilon j})_{j\in\mathbb{N}_0}$. In this proof we will denote by β the sequence $(a)\sigma = (2^{aj}\sigma_j)_{j\in\mathbb{N}_0}$. For all $j \in \mathbb{N}_0$, $l \in \mathbb{Z}^n$, by Theorem 3.18, Proposition 3.4 and Definition 3.21,

$$\|a^{j,l}(2^{-\varepsilon j}\cdot)|B_p^{T_j(\beta_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\| \sim 2^{\frac{\varepsilon j n}{p}} \|a^{j,l}|B_p^\beta(\mathbb{R}^n)\| \le 2^{\varepsilon j(a+\frac{n}{p})}. \tag{3.4.3}$$

By Theorem 3.14, for all $j \in \mathbb{N}_0$, $l \in \mathbb{Z}^n$, there is $\lambda^{j,l} \in b_p$ such that

$$a^{j,l}(2^{-\varepsilon j}x) = \sum_{k=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda^{j,l}_{k,m} b^{k,m}_{j,l}(x),$$

with

$$c\|\lambda^{j,l}|b_p\| \le \|a^{j,l}(2^{-\varepsilon j}\cdot)|B_p^{T_j(\beta_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|, \tag{3.4.4}$$

where the constant c is independent of j and $b^{k,m}_{j,l}$ are d-$(T_j(\beta_\varepsilon), p)_K$-$\varepsilon$-atoms located at $d'Q_{\varepsilon k,m}$, for some $d' > 1$ fixed, with $K > a + \overline{s}(\sigma)$.

The functions defined by

$$d^{j+k,m}_{j,l}(x) := 2^{\varepsilon a(j+k)} 2^{\frac{\varepsilon j n}{p}} b^{k,m}_{j,l}(2^{\varepsilon j}x)$$

are d-$(\sigma_\varepsilon, p)_K$-$\varepsilon$-atoms located at $d'Q_{\varepsilon(j+k),m}$.

Then

$$\begin{aligned} a^{j,l}(x) &= 2^{-\varepsilon j(a+\frac{n}{p})} \sum_{k=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{-\varepsilon a k} \lambda^{j,l}_{k,m} d^{j+k,m}_{j,l}(x) \\ &= 2^{-\varepsilon j(a+\frac{n}{p})} \sum_{k=j}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{-\varepsilon a(k-j)} \lambda^{j,l}_{k-j,m} d^{k,m}_{j,l}(x). \end{aligned}$$

Hence

$$f = \sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\sum_{k=j}^{\infty}\sum_{m\in\mathbb{Z}^n} 2^{-\varepsilon j\frac{n}{p}}2^{-\varepsilon ak}\nu_{j,l}\lambda^{j,l}_{k-j,m}d^{k,m}_{j,l}(x)\chi_{B(y^{j,l},d2^{-\varepsilon j})}(x)$$
$$= \sum_{k=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\sum_{j\le k}\sum_{l\in\mathbb{Z}^n} 2^{-\varepsilon j\frac{n}{p}}2^{-\varepsilon ak}\nu_{j,l}\lambda^{j,l}_{k-j,m}d^{k,m}_{j,l}(x)\chi_{B(y^{j,l},d2^{-\varepsilon j})}(x).$$

For $k\in\mathbb{N}_0, m\in\mathbb{Z}^n$ and $j\le k$ let (j,k,m) be the collection of all $l\in\mathbb{Z}^n$ such that $\operatorname{supp} a^{j,l}\cap\operatorname{supp} d^{k,m}_{j,l}$ is not empty. Each of such sets has at most M elements, where M is a natural number independent of j,k and m.Then, we get

$$f = \sum_{k=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\eta_{k,m}d^{k,m}(x),$$

where

$$\eta_{k,m} := \sum_{j\le k}\sum_{l\in(j,k,m)}|\nu_{j,l}|\cdot|\lambda^{j,l}_{k-j,m}|2^{-\varepsilon ak}2^{-\frac{\varepsilon jn}{p}}$$

and

$$d^{k,m}(x) := \frac{\sum_{j\le k}\sum_{l\in(j,k,m)}2^{-\frac{\varepsilon jn}{p}}\nu_{j,l}\lambda^{j,l}_{k-j,m}d^{k,m}_{j,l}(x)}{\sum_{j\le k}\sum_{l\in(j,k,m)}2^{-\frac{\varepsilon jn}{p}}|\nu_{j,l}|\cdot|\lambda^{j,l}_{k-j,m}|}.$$

are d-$(\sigma_\varepsilon,p)_K$-ε-atoms located at $d'Q_{\varepsilon k,m}$. We can prove that for some fixed $\delta\in(0,\varepsilon a)$, for all $0<p<\infty$,

$$|\eta_{k,m}|^p \le c2^{-\varepsilon akp}\sum_{j\le k}\sum_{l\in(j,k,m)}|\nu_{j,l}|^p\cdot|\lambda^{j,l}_{k-j,m}|^p2^{(k-j)\delta p}2^{-\varepsilon jn}.$$

Therefore, by (3.4.3) and (3.4.4), we obtain

$$\|\eta|b_p\|^p \le c\sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n}|\nu_{j,l}|^p2^{-\varepsilon j(n+ap)}\sum_{k=j}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda^{j,l}_{k-j,m}|^p \le c'\|\nu|b_p\|$$

and, by Theorem 3.14 and Proposition 3.4, the proof is complete. ∎

Remark 3.24. Let us have a closer look to the type of convergence of (3.4.2) in Theorem 3.23. If $\nu\in b_p$, then (3.4.2) converges in $L_{\overline{p}}(\mathbb{R}^n)$. This follows from the estimations obtained in Proposition 3.22. Let $0<p<\infty$ (the case $p=\infty$ is done with the usual modifications) and $\delta\in(0,\underline{s}(\sigma)\varepsilon)$. Making use of the controlled overlapping of the supports of the atoms, which follows from (3.2.1), we obtain

$$\Big\|\sum_{l\in\mathbb{Z}^n}\nu_{j,l}a^{j,l}|L_p(\mathbb{R}^n)\Big\|^p \le c\sum_{l\in\mathbb{Z}^n}|\nu_{j,l}|^p\|a^{j,l}|L_p(\mathbb{R}^n)\|^p \le c'2^{-(\underline{s}(\sigma)\varepsilon-\delta)jp}\sum_{l\in\mathbb{Z}^n}|\nu_{j,l}|^p.$$

Let

$$f_T := \sum_{j=0}^{T}\sum_{l\in\mathbb{Z}^n}\nu_{j,l}a^{j,l}.$$

Then for $T,M\in\mathbb{N}_0$, with $M<T$, and writing $\widetilde{p}=\min\{1,p\}$

$$\|f_T-f_M|L_p(\mathbb{R}^n)\|^{\widetilde{p}} \le \sum_{j=M+1}^{T}\Big\|\sum_{l\in\mathbb{Z}^n}\nu_{j,l}a^{j,l}|L_p(\mathbb{R}^n)\Big\|^{\widetilde{p}} \le c2^{-M(\underline{s}(\sigma)\varepsilon-\delta)\widetilde{p}}\|\nu|b_p\|^{\widetilde{p}}$$

and so $(f_T)_T$ converges in $L_p(\mathbb{R}^n)$. If $0 < p < 1$, then, for all admissible sequences σ,

$$B_p^{\sigma}(\mathbb{R}^n) \subset B_1^{\sigma(n)^{1-\frac{1}{p}}}(\mathbb{R}^n).$$

This follows from [3, p. 56. 2.2.16, 2.2.17]. So, if $a^{j,l}$ are d-$(\sigma, p)_a$-ε-atoms, they also are d-$(\sigma(n)^{1-\frac{1}{p}}, 1)_a$-$\varepsilon$-atoms and then, from the previous calculations, it follows that (3.4.2) converges in $L_1(\mathbb{R}^n)$.

4. Besov spaces of generalised smoothness on h-sets

Let us recall the definition and some properties of h-sets in $\mathbb{R}^n$ studied in [3]. As we have already said, we will rely on what is known about this kind of sets in $\mathbb{R}^n$ to extend this theory to more general spaces.

4.1. h-sets.

Definition 4.1. Let $\mathbb{H}$ denote the class of all continuous monotone increasing functions $h : (0,\infty) \to (0,\infty)$ such that $h(0^+) = 0$. We refer to $\mathbb{H}$ as the set of all *gauge functions.*

In what follows, for $h \in \mathbb{H}$ and $\alpha > 0$, we denote by $\boldsymbol{h}_\alpha$ the sequence

$$\boldsymbol{h}_\alpha := (h(2^{-\alpha j}))_{j \in \mathbb{N}_0}. \tag{4.1.1}$$

If $\alpha = 1$ we shall write only $\boldsymbol{h}$.

Definition 4.2. Let $h \in \mathbb{H}$ and Γ be a non-empty compact set of $\mathbb{R}^n$. We say that Γ is an *h-set* if there exists a finite Radon measure μ such that

$$\operatorname{supp} \mu = \Gamma$$

and

$$\mu(B(\gamma, r)) \sim h(r), \quad 0 < r \le \operatorname{Diam}\Gamma,\ \gamma \in \Gamma.$$

Then we say that h is a *measure function* (in $\mathbb{R}^n$) and that μ is an *h-measure* (related to Γ).

Remark 4.3. If the function h is given by

$$h(r) = r^d \psi(r), \quad 0 < r \le 1,$$

where $0 < d < n$ and $\psi : (0, 1] \to \mathbb{R}^+$ is a monotone function such that

$$\psi(2^{-j}) \sim \psi(2^{-2j}), \quad \text{for all } j \in \mathbb{N}_0,$$

then we say that Γ is a *(d, ψ)-set.* These sets were introduced by Edmunds and Triebel in [9] and studied by Moura in [20, 19]. If, additionally, $\psi \sim 1$ then we say that Γ is a *d-set.* Besov spaces with classical smoothness on d-sets have been studied by many authors, in particular by Jonsson and Wallin in [15] and by Triebel in [25] and [26].

Remark 4.4. By [3, Theorem 1.7.6, p. 22], if Γ is an h-set, then all h-measures related to Γ are equivalent to $\mathcal{H}^h_\Gamma$, where $\mathcal{H}^h_\Gamma$ is the restriction of the Hausdorff measure $\mathcal{H}^h_\Gamma$ in $\mathbb{R}^n$ to Γ.

Bricchi characterised in [4] which functions h are measure functions with the following outcome.

THEOREM 4.5. *Let $h \in \mathbb{H}$. Then h is a measure function in $\mathbb{R}^n$ if, and only if, there exists a gauge function $\widetilde{h} \sim h$ such that*

$$\frac{\tilde{h}(2^{-(j+k)})}{\tilde{h}(2^{-j})} \geq 2^{-kn}, \quad j,k \in \mathbb{N}_0.$$

We now present a definition about a geometric property of sets. It is useful when working with traces on Besov spaces on $\mathbb{R}^n$.

DEFINITION 4.6. A non-empty Borel set Γ satisfies the *ball condition* (or *porosity condition*) if there exists a number $0 < \eta < 1$ with the following property:
for any ball $B(\gamma, r)$ with $\gamma \in \Gamma$ and $0 < r \leq 1$ there is a ball $B(x, \eta r)$ centred at $x \in \mathbb{R}^n$ such that

$$B(x, \eta\, r) \subset B(\gamma, r) \quad \textit{and} \quad B(x, \eta r) \cap \overline{\Gamma} = \emptyset.$$

The next theorem can be found in [26, pp. 139/140, Proposition 9.18].

THEOREM 4.7. *Let $\Gamma \subset \mathbb{R}^n$ be an h-set. Then Γ satisfies the ball condition if, and only if, there are two positive constants c and δ such that*

$$h(2^{-\nu}) \leq c\, 2^{(n-\delta)\varkappa} h(2^{-\nu-\varkappa}), \quad \textit{for all}\ \nu, \varkappa \in \mathbb{N}_0.$$

COROLLARY 4.8. *Let $\Gamma \subset \mathbb{R}^n$ be an h-set. Then Γ satisfies the ball condition if, and only if,*

$$\underline{s}(\boldsymbol{h}) > -n, \tag{4.1.2}$$

with $\underline{s}(\boldsymbol{h})$ according to Definitions 2.9 and (4.1.1).

REMARK 4.9. If $h(r) = r^d$, $r > 0$, then (4.1.2) is equivalent to $d < n$.

DEFINITION 4.10. Let Γ be an h-set and let us fix an admissible sequence σ. Let $0 < p, q < \infty$. Suppose that there exists a positive constant c such that

$$\|\varphi|_\Gamma \,|\, L_p(\Gamma)\| \leq c\, \|\varphi \,|\, B^\sigma_{p,q}(\mathbb{R}^n)\|, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{4.1.3}$$

Let us consider $f \in B^\sigma_{p,q}(\mathbb{R}^n)$. As $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^\sigma_{p,q}(\mathbb{R}^n)$, there is a sequence $\{\varphi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ such that

$$\varphi_j \to f, \quad \text{as } j \to \infty, \quad \text{in } B^\sigma_{p,q}(\mathbb{R}^n).$$

By (4.1.3) the sequence $\{\varphi_j|_\Gamma\}_{j\in\mathbb{N}_0}$ converges in $L_p(\Gamma)$ to an element which we call *trace of f* and we denote by $\mathrm{tr}_\Gamma\, f$.

DEFINITION 4.11. Consider an h-set $\Gamma \subset \mathbb{R}^n$ satisfying the ball condition. Let σ be an admissible sequence, with $\underline{s}(\sigma) > 0$ and let $0 < p, q \leq \infty$. Then we define

$$\mathbb{B}^\sigma_{p,q}(\Gamma) = \mathrm{tr}_\Gamma\ B^{\sigma \boldsymbol{h}^{1/p} (n)^{1/p}}_{p,q}(\mathbb{R}^n) \tag{4.1.4}$$

endowed with the quasi-norm

$$\|f \,|\, \mathbb{B}^\sigma_{p,q}(\Gamma)\| = \inf \|g \,|\, B^{\sigma \boldsymbol{h}^{1/p} (n)^{1/p}}_{p,q}(\mathbb{R}^n)\|,$$

where the infimum is taken over all $g \in B^{\sigma\, \boldsymbol{h}^{1/p} (n)^{1/p}}_{p,q}(\mathbb{R}^n)$ such that $\mathrm{tr}_\Gamma\, g = f$.
If $p = q$ we denote these spaces by $\mathbb{B}^\sigma_p(\Gamma)$.

Remark 4.12. We recall that

$$\sigma\, \boldsymbol{h}^{1/p}(n)^{1/p} = (\sigma_j h(2^{-j})^{\frac{1}{p}} 2^{\frac{jn}{p}})_{j\in\mathbb{N}_0}.$$

The above definition was given in [3, Chapter 3], where Bricchi showed that the definition makes sense and that, if we apply it to $\sigma = (0)$, we get that

$$\mathbb{B}^{(0)}_{p,q}(\Gamma) = L_p(\Gamma), \quad 0 < p < \infty, \quad 0 < q \leq \min\{1,p\}.$$

In Definition 4.10, tr_Γ was defined just for $0 < p, q < \infty$. But, if $\underline{s}(\sigma) > 0$,

$$B^{\sigma \boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n) \subset B^{\boldsymbol{h}^{1/p}(n)^{1/p}}_{p,\min(1,p)}(\mathbb{R}^n)$$

and the trace is well-defined in the space on the right, so the above definition also makes sense for $q = \infty$ and $0 < p < \infty$. If $p = \infty$ and $\underline{s}(\sigma) > 0$ then

$$B^{\sigma}_{\infty,q}(\mathbb{R}^n) \subset B^{(0)}_{\infty,1}(\mathbb{R}^n) \subset \mathcal{C}(\mathbb{R}^n),$$

where $\mathcal{C}(\mathbb{R}^n)$ is the space of all bounded and uniformly continuous functions in $\mathbb{R}^n$, normed in the usual way. Hence, the trace of $f \in B^{\sigma}_{\infty,q}(\mathbb{R}^n)$ is defined as the pointwise restriction. Bricchi uses of the letter "$\mathbb{B}$", following the notation used by Triebel in [25] for Besov spaces on d-sets. In [25] in the definition of d-set it is not assumed that the set is compact. So, if $d, n \in \mathbb{N}$ and $d < n$, $\mathbb{R}^d$ is a d-set in $\mathbb{R}^n$. So there was already a definition for Besov spaces on $\mathbb{R}^d$ that, in some conditions, do not coincide with

$$\mathbb{B}^{(s)}_{p,q}(\mathbb{R}^d) = \mathrm{tr}_{\mathbb{R}^d}\ B^{\left(s+\frac{n-d}{p}\right)}_{p,q}(\mathbb{R}^n).$$

More details about this may be found in [25, p. 160].

Example 4.13. Let us consider the following particular case where we have Besov spaces on (d,ψ)-sets, according to Remark 4.3. So we have

$$h(r) \sim r^d \psi(r), \quad 0 < r \leq \operatorname{Diam}\Gamma,$$

and

$$\sigma_j = 2^{js}\psi(2^{-j})^a, \quad \text{for any } j \in \mathbb{N}_0.$$

In [20], Moura defined

$$\mathbb{B}^{(s,\psi^a)}_{p,q}(\Gamma) = \mathrm{tr}_\Gamma\ B^{(s+\frac{n-d}{p},\psi^{1/p+a})}_{p,q}(\mathbb{R}^n).$$

This case is included in Definition 4.11. We have

$$\sigma \boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}} = \left(2^{js}\psi(2^{-j})^a 2^{-\frac{jd}{p}}\psi(2^{-j})^{\frac{1}{p}} 2^{\frac{jn}{p}}\right)_{j\in\mathbb{N}_0} = \left(s + \frac{n-d}{p},\ \psi^{\frac{1}{p}+a}\right).$$

If we consider additionally $\psi \sim 1$ then we get Besov spaces with classical smoothness on d-sets.

4.2. Characterisation by atomic decompositions. Let Γ be a compact set in $\mathbb{R}^n$ and $\delta > 0$. Then

$$\Gamma_\delta = \{x \in \mathbb{R}^n : dist\,(x,\Gamma) < \delta\}$$

denotes a δ-neighbourhood of Γ.

DEFINITION 4.14. Let $0 < \varepsilon \le 1$ and $j \in \mathbb{N}_0$. We say that

$$\{\gamma^{j,m} : m = 1, ..., M_j\} \subset \Gamma \tag{4.2.1}$$

are $2^{-\varepsilon j}$-*approximate lattices* for Γ if there exist positive numbers $c_{\varepsilon,1}$, $c_{\varepsilon,2}$ and $c_{\varepsilon,3}$ with

$$|\gamma^{j,m_1} - \gamma^{j,m_2}| \ge c_{\varepsilon,1}\, 2^{-\varepsilon j}, j \in \mathbb{N}_0, m_1 \neq m_2, \tag{4.2.2}$$

and

$$\Gamma_{\delta_j} \subset \bigcup_{m=1}^{M_j} B(\gamma^{j,m}, c_{\varepsilon,2}2^{-\varepsilon j}), \quad j \in \mathbb{N}_0, \tag{4.2.3}$$

where $\delta_j = c_{\varepsilon,3}2^{-\varepsilon j}$.

REMARK 4.15. If Γ is an h-set, then, for $j \in \mathbb{N}_0$,

$$M_j \sim h(2^{-\varepsilon j})^{-1}.$$

This can be proved applying [3, p. 30, Lemma 1.8.3].

The approximate lattices for Γ, $\{\gamma^{j,m}\}_{m=1}^{M_j}$, $j \in \mathbb{N}_0$, can be extended to approximate lattices in $\mathbb{R}^n$ as in Definition 3.6.
So, using the notation of the above mentioned definition, we have that for any $j \in \mathbb{N}_0$ there is $L_j = \{l_{j,1}, ..., l_{j,M_j}\} \subset \mathbb{Z}^n$ such that

$$y^{j,l_{j,m}} = \gamma^{j,m}, \quad m = 1, ..., M_j.$$

ASSUMPTION 4.16. Let $0 < \varepsilon \le 1$ and Γ be an h-set in $\mathbb{R}^n$. For all $j \in \mathbb{N}_0$, we will denote by

$$\{\gamma_{j,m}\}_{m=1}^{M_j} \quad \text{and} \quad \{\delta_{j,t}\}_{t=1}^{T_j}$$

$2^{-\varepsilon j}$ and 2^{-j}-approximate lattices, respectively, for Γ, according to (4.2.1)-(4.2.3). In what follows, in all results involving approximate lattices, we assume that they and their extensions to corresponding approximate lattices in $\mathbb{R}^n$ have been fixed.

We have to adapt previous definitions:

DEFINITION 4.17. Let $0 < p, q \le \infty$ and

$$\lambda = \{\lambda_{j,m} \in \mathbb{C} : j \in \mathbb{N}_0, m = 1, ..., R_j\}$$

Then we define

$$b_{p,q}^{\Gamma} = \Big\{\lambda : \|\lambda \,|\, b_{p,q}^{\Gamma}\| = \Big(\sum_{j=0}^{\infty}\Big(\sum_{m=1}^{R_j}|\lambda_{j,m}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} < \infty\Big\},$$

with the usual modification if $p = \infty$ or $q = \infty$ and with the abbreviation b_p^{Γ} if $p = q$.

DEFINITION 4.18. Let $\Gamma \subset \mathbb{R}^n$ be an h-set fulfilling the ball condition. Let σ be an admissible sequence, $0 < \varepsilon \le 1$, $0 < p \le \infty$, $K \in \mathbb{N}_0$ and $d > c_{\varepsilon,2}$. Then a function $a \in \mathcal{C}^K(\mathbb{R}^n)$ is called a d-$(\sigma, p)_K^{\Gamma}$-ε-atom if

(a) $\operatorname{supp} a \subset B(\gamma^{j,m}, d2^{-\varepsilon j})$, for some $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$;

(b) $\sup_{x\in\mathbb{R}^n} |D^\alpha a(x)| \le \sigma_j^{-1}\, h(2^{-\varepsilon j})^{-\frac{1}{p}}\, 2^{|\alpha|\varepsilon j}, \qquad \text{for } |\alpha| \le K.$

Bricchi obtained decompositions with this kind of atoms for the elements of Besov spaces on h-sets in the particular case where $\varepsilon = 1$ and taking atoms located in cubes $dQ_{j,m}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $d > 1$ (cf. [3, p. 117]). The following theorem is just an adaptation of that result, obtained from an analogous characterisation for Besov spaces on $\mathbb{R}^n$ presented in Theorem 3.14.

THEOREM 4.19. *Let $\Gamma \subset \mathbb{R}^n$ be an h-set. Let σ be an admissible sequence, $0 < p, q \le \infty$ and $0 < \varepsilon \le 1$. We consider*

$$\underline{s}(\boldsymbol{h}) > -n \quad \text{and} \quad \underline{s}(\sigma) > -\underline{s}(\boldsymbol{h})\Big(\frac{1}{p} - 1\Big)_+. \tag{4.2.4}$$

Let $d > c_{\varepsilon,2}$ and $K > \overline{s}(\sigma \boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}})$. Then $\mathbb{B}^\sigma_{p,q}(\Gamma)$ is the collection of all $f \in L_p(\Gamma)$ such that

$$f = \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \nu_{j,m} a^{j,m}(x), \qquad \text{in } L_p(\Gamma), \tag{4.2.5}$$

for some $\nu \in b^\Gamma_{p,q}$ and some family of d-$(\sigma_\varepsilon, p)^\Gamma_K$-$\varepsilon$-atoms $a^{j,m}$. Furthermore,

$$\|f|\mathbb{B}^\sigma_{p,q}(\Gamma)\| \sim \inf \|\nu|b^\Gamma_{p,q}\|,$$

where the infimum is taken over all representations (4.2.5).

REMARK 4.20. Condition (4.2.4) is used to guarantee that

$$\underline{s}(\sigma \boldsymbol{h}^{1/p}(n)^{1/p}) > n\Big(\frac{1}{p} - 1\Big)_+,$$

so that Theorem 3.14 can be applied to the spaces $B^{\sigma \boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n)$. This possibility and also (4.1.4) were used by Bricchi to obtain Theorem 4.19.
As it was mentioned in Corollary 4.8, the assumption $\underline{s}(\boldsymbol{h}) > -n$ implies that Γ fulfills the ball condition.

Next, we present the definition of what we call non-smooth atoms (on a compact set). As we shall see later, it is convenient for us to consider two kinds of non-smooth atoms. Recall the notation introduced in Assumption 4.16.

DEFINITION 4.21. Let $\Gamma \subset \mathbb{R}^n$ be a an h-set, $0 < \varepsilon \le 1$ and $d > c_{\varepsilon,2}$, where $c_{\varepsilon,2}$ is as in (4.2.3). Let $0 < p \le \infty$ and σ be an admissible sequence. Then a Lipschitz-continuous function $a^{j,m}_\Gamma$ on Γ is called a *d-$(\sigma, p)^*_\Gamma$-ε-atom* if for $j \in \mathbb{N}_0$, $m = 1, ..., M_j$,
(a) $\operatorname{supp} a^{j,m}_\Gamma \subset B(\gamma^{j,m}, d2^{-\varepsilon j}) \cap \Gamma,$

(b) $|a^{j,m}_\Gamma(\gamma)| \le \sigma_j^{-1}\, h(2^{-\varepsilon j})^{-\frac{1}{p}}, \qquad \gamma \in \Gamma,$

(c) $|a^{j,m}_\Gamma(\gamma) - a^{j,m}_\Gamma(\delta)| \le \sigma_j^{-1}\, h(2^{-\varepsilon j})^{-\frac{1}{p}} 2^{\varepsilon j}|\gamma - \delta|, \qquad \gamma, \delta \in \Gamma.$

DEFINITION 4.22. Let Γ, ε, p and σ be as in Definition 4.21. Let $d > c_{1,2}$ where $c_{1,2}$ is as in (4.2.3) for the 2^{-j}-approximate lattices $\{\delta^{j,t}\}_t$ (with $\varepsilon = 1$). A Lipschitz-continuous function $a^{j,t}_\Gamma$ on Γ is called a *d-$(\sigma, p, \varepsilon)^{**}_\Gamma$-atom* if for $j \in \mathbb{N}_0$, $t = 1, ..., T_j$,

(a) $\operatorname{supp} a_\Gamma^{j,t} \subset B(\delta^{j,t}, d2^{-j}) \cap \Gamma$,

(b) $|a_\Gamma^{j,t}(\gamma)| \leq \sigma_j^{-1}\, h(2^{-j})^{-\frac{1}{p}}, \qquad \gamma \in \Gamma$,

(c) $|a_\Gamma^{j,t}(\gamma) - a_\Gamma^{j,t}(\delta)| \leq \sigma_j^{-1}\, h(2^{-j})^{-\frac{1}{p}} 2^{\varepsilon j} |\gamma - \delta|^\varepsilon, \qquad \gamma, \delta \in \Gamma$.

Let $h \in \mathbb{H}$ and $\Gamma \subset \mathbb{R}^n$ be an h-set. Assume that

$$-\underline{s}(\boldsymbol{h}) < n \qquad \text{and} \qquad -\overline{s}(\boldsymbol{h}) > 0, \tag{4.2.6}$$

which corresponds to the condition $0 < d < n$ in the particular case of d-sets. According to Corollary 4.8, Γ will then fulfill the ball condition. Let $\delta \in (0, -\overline{s}(\boldsymbol{h}))$. By Definition 2.9, there exist positive numbers c_δ and c'_δ such that

$$c_\delta \lambda^{-\underline{s}(\boldsymbol{h})+\delta} \leq \frac{h(\lambda t)}{h(t)} \leq c'_\delta \lambda^{-\overline{s}(\boldsymbol{h})-\delta}, \qquad 0 < \lambda, t \leq 1. \tag{4.2.7}$$

This kind of condition was considered in [14] and [17]. In these papers Besov spaces on h-sets were defined following the same kind of approach of Jonsson and Wallin in [15] for d-sets. They proved the existence of an extension operator for convenient spaces on $\mathbb{R}^n$ and of a restriction operator from these spaces back to function spaces on h-sets. In [14], Jonsson considered Besov spaces on h-sets which are the trace of spaces on $\mathbb{R}^n$ with classical smoothness, i.e., spaces

$$\mathbb{B}_p^{(s)\boldsymbol{h}^{-\frac{1}{p}}(n)^{-\frac{1}{p}}}(\Gamma) = \operatorname{tr}_\Gamma B_p^{(s)}(\mathbb{R}^n),$$

where s is a positive real number satisfying certain conditions in connection with (4.2.7) (cf. [14, p.357, Theorem 1]). In [17], Knopova and Zähle considered spaces of generalised smoothness

$$\mathbb{B}_p^{\tau\boldsymbol{h}^{-\frac{1}{p}}(n)^{-\frac{1}{p}}}(\Gamma) = \operatorname{tr}_\Gamma B_p^{\tau}(\mathbb{R}^n),$$

with $\tau_j = f(2^{2j})^{\frac{\alpha}{2}}$, $j \in \mathbb{N}_0$, where f are Bernstein functions satisfying a list of conditions and α satisfies conditions also related to (4.2.7) (cf. [17, Theorem 18]). Most of the conditions considered for the functions f were applied to prove the existence and continuity of the restriction operator.

For our purposes it is convenient not to have so strong conditions for the class of sequences considered. Furthermore, we only need to work with extension operators acting in a class of fractals obtained as dilations of a fixed h-set and taking sequences $T_k(\sigma)$ for the smoothness. So, we will take extension operators defined analogously but acting in a larger scale of function spaces.

Assumption 4.23. Let $h \in \mathbb{H}$ and $\Gamma \subset \mathbb{R}^n$ be an h-set. We consider $c_1, c_2 > 0$ such that

$$c_1 h(r) \leq \mu(B(\gamma, r)) \leq c_2 h(r), \quad \gamma \in \Gamma, \quad 0 < r \leq 1. \tag{4.2.8}$$

We assume that h satisfies (4.2.6) and, consequently, (4.2.7).

Let, for $r > 0$,

$$B^\Gamma(r) = \{\gamma \in \Gamma : |\gamma - \gamma^0| < r\}, \quad \text{for some} \quad \gamma^0 \in \Gamma. \tag{4.2.9}$$

For $0 < \varepsilon \le 1$ and $k \in \mathbb{N}_0$, let

$$D_{\varepsilon k} : x \mapsto 2^{\varepsilon k} x, \quad x \in \mathbb{R}^n.$$

We define

$$\mathbf{F}_{\varepsilon k} := D_{\varepsilon k}\Gamma$$

and

$$\Gamma_{\varepsilon k} := D_{\varepsilon k} B^{\Gamma}(2c_0 2^{-\varepsilon k}).$$

We consider the image measure

$$\mu^{\varepsilon k} := \mu \circ D_{\varepsilon k}^{-1}.$$

Set also

$$\mu_{\varepsilon k} := \frac{\mu^{\varepsilon k}}{h(2^{-\varepsilon k})} \quad \text{and} \quad h^*_{\varepsilon,k}(r) := \frac{h(2^{-\varepsilon k} r)}{h(2^{-\varepsilon k})}, \quad r > 0.$$

If $\widetilde{\gamma} = 2^{\varepsilon k}\gamma$ then, for all $r > 0$,

$$\mu^{\varepsilon k}(B(\widetilde{\gamma}, r)) = \mu(B(\gamma, 2^{-\varepsilon k} r)).$$

Therefore

$$c_1 h^*_{\varepsilon,k}(r) \le \mu_{\varepsilon k}(B(\widetilde{\gamma}, r)) \le c_2 h^*_{\varepsilon,k}(r), \quad \widetilde{\gamma} \in \mathbf{F}_{\varepsilon k}, \quad 0 < r \le 1,$$

where c_1 and c_2 are the same as in (4.2.8) and, so, independent of k.
There are $c_3, c_4 > 0$, also independent of k, such that

$$c_3 \le \mu_{\varepsilon k}(\Gamma_{\varepsilon k}) \le c_4. \tag{4.2.10}$$

Moreover, (4.2.7) implies that

$$c_\delta \lambda^{-\underline{s}(\boldsymbol{h})+\delta} \le \frac{h^*_{\varepsilon,k}(\lambda t)}{h^*_{\varepsilon,k}(t)} \le c'_\delta \lambda^{-\overline{s}(\boldsymbol{h})-\delta}, \qquad 0 < \lambda, t \le 1.$$

Definition 4.24. Consider the conditions described in Assumption 4.23. Let $0 < \varepsilon \le 1$, $N_\varepsilon = (2^{\varepsilon j})_{j \in \mathbb{N}_0}$, $1 \le p \le \infty$, σ be an admissible sequence and Λ be an admissible function associated to σ. We denote by $\widetilde{B}_p^{T_k(\sigma_\varepsilon), N_\varepsilon}(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})$ the collection of all $\widetilde{u}$ such that

$$\widetilde{u} \in L_p(\mathbf{F}_{\varepsilon k}, \mu_{\varepsilon k}), \quad \operatorname{supp} \widetilde{u} \subset D_{\varepsilon k} B^{\Gamma}(c_0 2^{-\varepsilon k})$$

and, for $1 \le p < \infty$,

$$\begin{aligned} \|\widetilde{u}|\widetilde{B}_p^{T_k(\sigma_\varepsilon), N_\varepsilon}(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})\| \; &:= \; \sigma_{\varepsilon,k} \|\widetilde{u}|L_p(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})\| \\ &+ \left(\int_{\Gamma_{\varepsilon k}} \int_{\Gamma_{\varepsilon k}} \frac{\Lambda(2^{\varepsilon k}|t-v|^{-1})^p}{h^*_{\varepsilon,k}(|t-v|)} |\widetilde{u}(t) - \widetilde{u}(v)|^p d\mu_{\varepsilon k}(t) d\mu_{\varepsilon k}(v) \right)^{1/p} \end{aligned}$$

is finite, or, for $p = \infty$,

$$\|\widetilde{u}|\widetilde{B}_\infty^{T_k(\sigma_\varepsilon), N_\varepsilon}(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})\| := \sigma_{\varepsilon,k} \|\widetilde{u}|L_\infty(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})\| + \sup_{t,v \in \Gamma_{\varepsilon k}} \Lambda(2^{\varepsilon k}|t-v|^{-1}) |\widetilde{u}(t) - \widetilde{u}(v)|$$

is finite.

In what follows by *cube* we mean a closed cube in $\mathbb{R}^n$, with sides parallel to the axes.

Definition 4.25. We will consider $\mathcal{Q} = \{Q_i\}_i$ a numerable collection of cubes such that

(i) $\bigcup_i Q_i = \mathbb{R}^n \setminus \mathbf{F}_{\varepsilon k}$,
(ii) The interiors of the cubes Q_i are mutually disjoint,
(iii) $\operatorname{diam}(Q_i) \le \operatorname{dist}(Q_i, \mathbf{F}_{\varepsilon k}) \le 4 \operatorname{diam}(Q_i)$.

We assume that for all $Q_i \in \mathcal{Q}$ there is $j \in \mathbb{Z}$ such that the side length of Q_i is 2^{-j}.

REMARK 4.26. We refer to [22, p. 167-170] where we can find a proof of the existence of such a decomposition.

We fix $0 < \eta < 1/4$, which is arbitrary but will be kept fixed in what follows. We denote by Q_i^* the cube with the same center as Q_i but expanded by the factor $(1+\eta)$, i.e., $Q_i^* = (1+\eta)Q_i$.

In [22] the following result was also proved.

PROPOSITION 4.27. *There exists a numerable collection of functions* $\{\varphi_i\}_i \subset C_0^\infty(\mathbb{R}^n)$ *such that*

(i) $\varphi_i(x) = 0, \quad$ *if* $x \notin Q_i^*$,
(ii) $\sum_i \varphi_i(x) = 1, \quad$ *if* $x \in \mathbb{R}^n \setminus \mathbf{F}_{\varepsilon k}$,
(iii) $|D^\alpha \varphi_i(x)| \le A_\alpha (\operatorname{diam} Q_i)^{-|\alpha|}, \quad x \in \mathbb{R}^n, \quad \alpha \in {\mathbb{N}_0}^n$.

In what follows, given a cube Q_i we will denote its center by x_i, its side length by s_i and its diameter by l_i.

We introduce some more notation:

$$C_i := (\mu_{\varepsilon k}(B(x_i, 6l_i)))^{-1}$$

DEFINITION 4.28. Let

$$I := \{i : s_i \le 1\}.$$

We define, for $\widetilde{u} \in \widetilde{B}_p^{T_k(\sigma_\varepsilon), N_\varepsilon}(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})$,

$$\mathscr{E}_{\varepsilon,k}\widetilde{u}(x) := \sum_{i \in I} \varphi_i(x) C_i \int_{|t-x_i| \le 6l_i} \widetilde{u}(t) d\mu_{\varepsilon k}(t), \quad x \in \mathbb{R}^n \setminus \mathbf{F}_{\varepsilon k}. \tag{4.2.11}$$

REMARK 4.29. By the conditions on the support of $\widetilde{u}$, we can replace in the integral in (4.2.11)

$$\int_{|t-x_i| \le 6l_i} \quad \text{by} \quad \int_{t \in \Gamma_{\varepsilon k},\ |t-x_i| \le 6l_i}$$

REMARK 4.30. Let σ be an admissible sequence and Λ be an admissible function associated to σ. Let a and A be given by

$$a := \sigma \boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}} \quad \text{and} \quad A(x) = \Lambda(x) h(x^{-1})^{\frac{1}{p}} x^{\frac{n}{p}}. \tag{4.2.12}$$

Then a is an admissible sequence and A is an admissible function associated to a.

THEOREM 4.31. *Let* $0 < \varepsilon \le 1$ *and* $N_\varepsilon = (2^{\varepsilon j})_{j \in \mathbb{N}_0}$. *Consider the conditions described in Assumption 4.23. Let* $1 \le p \le \infty$ *and* σ *be an admissible sequence such that* $\underline{s}(\sigma) > 0$. *Fix an admissible function* Λ *associated to* σ *and consider* a *and* A *as given by* (4.2.12) *and* $M \in \mathbb{N}$ *such that* $\overline{s}(a) < M$. *Let* $\widetilde{u} \in \widetilde{B}_p^{T_k(\sigma_\varepsilon), N_\varepsilon}(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})$. *Then* $\mathscr{E}_{\varepsilon,k}\widetilde{u} \in L_p(\mathbb{R}^n)$ *and there is* $c > 0$ *such that*

$$\begin{aligned}\|\mathscr{E}_{\varepsilon,k}\widetilde{u}|B_p^{T_k(a_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|_M &:= a_{\varepsilon,k}\|\mathscr{E}_{\varepsilon,k}\widetilde{u}|L_p(\mathbb{R}^n)\| \\ &+ \left(\int_{|u|\le R}\left(A(2^{\varepsilon k}|u|^{-1})\|\Delta_u^M(\mathscr{E}_{\varepsilon,k}\widetilde{u})|L_p(\mathbb{R}^n)\|\right)^p\frac{du}{|u|^n}\right)^{\frac{1}{p}}\end{aligned}$$

can be estimated from above by

$$ch(2^{-\varepsilon k})^{\frac{1}{p}}2^{\frac{n\varepsilon k}{p}}\|\widetilde{u}|\widetilde{B}_p^{T_k(\sigma_\varepsilon),N_\varepsilon}(\Gamma_{\varepsilon k},\mu_{\varepsilon k})\|,$$

where c is independent of k and γ^0 in (4.2.9). Furthermore,

$$(\mathscr{E}_{\varepsilon,k}\widetilde{u})_{|\mathbf{F}_{\varepsilon k}} = \widetilde{u}. \tag{4.2.13}$$

Remark 4.32. This assertion follows from the proofs of the *Extension Theorems* in [15, pp. 109-119] and [14, pp. 360-364], making convenient modifications.

Remark 4.33. Let $g = \mathscr{E}_{\varepsilon,k}\widetilde{u}$. By (4.2.13) we mean that for $\mu_{\varepsilon k}$-almost all $t_0 \in \mathbf{F}_{\varepsilon k}$

$$\lim_{r\to 0}\frac{1}{|B(t_0,r)|}\int_{B(t_0,r)} g(x)dx = \widetilde{u}(t_0). \tag{4.2.14}$$

We can easily check that

$$B_p^{T_k(a_\varepsilon),N_\varepsilon}(\mathbb{R}^n) =_k B_p^{T_{[\varepsilon k]}(\sigma)(\boldsymbol{h}^*_{\varepsilon,k})^{\frac{1}{p}}(n)^{\frac{1}{p}}}(\mathbb{R}^n),$$

where we are writing $=_k$ to make clear that in this equality we mean equivalent norms with constants depending of k. In fact, for all $f \in B_p^{T_k(a_\varepsilon),N_\varepsilon}(\mathbb{R}^n)$,

$$\|f|B_p^{T_k(a_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\| \sim 2^{\frac{n\varepsilon k}{p}}h(2^{-\varepsilon k})^{\frac{1}{p}}\|f|B_p^{T_{[\varepsilon k]}(\sigma)(\boldsymbol{h}^*_{\varepsilon,k})^{\frac{1}{p}}(n)^{\frac{1}{p}}}(\mathbb{R}^n)\|,$$

where the equivalence constants are now independent of k and where we applied (3.1.1). So, we can adapt the arguments in the proof of Theorem 3.4.15, pp. 114-116, of [3], to conclude that, in these conditions, for $1 < p < \infty$,

$$\mathrm{tr}_{\mathbf{F}_{\varepsilon k}}(\mathscr{E}_{\varepsilon,k}\widetilde{u}) = \widetilde{u}, \tag{4.2.15}$$

where $\mathrm{tr}_{\mathbf{F}_{\varepsilon k}}$ is given by Definition 4.10. If $p = \infty$, (4.2.15) follows immediately from (4.2.14), because in this case the trace is the pointwise restriction.

Theorem 4.34. *Let $\Gamma \subset \mathbb{R}^n$ be an h-set. Let $0 < \varepsilon \le 1$, $1 < p \le \infty$ and β be an admissible sequence. Suppose that*

$$-n < \underline{s}(\boldsymbol{h}) \le \overline{s}(\boldsymbol{h}) < 0.$$

(i) Let $d > c_{\varepsilon,2}$ and

$$0 < \underline{s}(\beta) \le \overline{s}(\beta) < 1.$$

Then $\mathbb{B}_p^\beta(\Gamma)$ is the collection of all $f \in L_p(\Gamma)$ such that

$$f = \sum_{k=0}^{\infty}\sum_{m=1}^{M_k}\nu_{k,m}a_\Gamma^{k,m}(\gamma), \qquad in\ L_p(\Gamma), \tag{4.2.16}$$

for some $\nu \in b_p^\Gamma$ and some family of d-$(\beta_\varepsilon,p)^*_\Gamma$-$\varepsilon$-atoms, $a_\Gamma^{k,m}$. Furthermore,

$$\|f|\mathbb{B}_p^\beta(\Gamma)\| \sim \inf\|\nu|b_p^\Gamma\|,$$

where the infimum is taken over all representations (4.2.16)*.*

(ii) *Let* $d > c_{1,2}$ *and*

$$0 < \underline{s}(\beta) \leq \overline{s}(\beta) < \varepsilon.$$

Then a corresponding result is true taking d*-*$(\beta, p, \varepsilon)^{**}_{\Gamma}$*-atoms (with* T_k *instead of* M_k *in* (4.2.16)*).*

Proof. Step 1. We will only present the proof for d-$(\beta_\varepsilon, p)^*_\Gamma$-$\varepsilon$-atoms, because the proof for d-$(\beta, p, \varepsilon)^{**}_\Gamma$-atoms is analogous. It follows immediately that all $f \in \mathbb{B}^\sigma_p(\Gamma)$ admits a representation (4.2.16) with $\|\nu|b^\Gamma_p\| \lesssim \|f|\mathbb{B}^\beta_p(\Gamma)\|$, because the restriction to Γ of d-$(\beta_\varepsilon, p)_K$-ε-atoms are special d-$(\beta_\varepsilon, p)^*_\Gamma$-$\varepsilon$-atoms.

Step 2. Let $f \in L_p(\Gamma)$ be as in (4.2.16), with $\nu \in b^\Gamma_p$. Let us prove that $f \in \mathbb{B}^\beta_p(\Gamma)$ and

$$\|f|\mathbb{B}^\beta_p(\Gamma)\| \lesssim \|\nu|b^\Gamma_p\|.$$

Let $k \in \mathbb{N}_0$, $m \in \{1, ..., M_k\}$ and $a^{k,m}_\Gamma$ be a d-$(\beta_\varepsilon, p)^*_\Gamma$-$\varepsilon$-atom located in $B(\gamma^{k,m}, d2^{-\varepsilon k})$, according to Definition 4.21. Following the notation used in Assumption 4.23, consider $\gamma^0 = \gamma^{k,m}$ and $c_0 = d$. Recall that

$$\mathbf{F}_{\varepsilon k} = D_{\varepsilon k}\Gamma \quad \text{and} \quad \Gamma_{\varepsilon k} = D_{\varepsilon k}(B(\gamma^{k,m}, 2d2^{-\varepsilon k}) \cap \Gamma) = \mathbf{F}_{\varepsilon k} \cap B(2^{\varepsilon k}\gamma^{k,m}, 2d).$$

Let $\alpha \in (0, 1 - \overline{s}(\beta))$, $\sigma = (\alpha)\beta$ and Λ be an admissible function associated to σ. Then

$$0 < \alpha + \underline{s}(\beta) = \underline{s}(\sigma) \leq \overline{s}(\sigma) = \alpha + \overline{s}(\beta) < 1$$

To simplify the notation, let us temporarily denote by $\widetilde{u}$ the function $a^{k,m}_\Gamma(2^{-\varepsilon k}\cdot)$. Then

$$\widetilde{u} \in L_p(\mathbf{F}_{\varepsilon k}, \mu_{\varepsilon k}), \quad \text{and} \quad \operatorname{supp}\widetilde{u} \subset D_{\varepsilon k}B^\Gamma(d2^{-\varepsilon k}).$$

Consider $N_\varepsilon = (2^{\varepsilon j})_{j\in\mathbb{N}_0}$. Let us prove that

$$\|\widetilde{u}|\widetilde{B}^{T_k(\sigma_\varepsilon),N_\varepsilon}_p(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})\| \leq ch(2^{-\varepsilon k})^{-\frac{1}{p}},$$

where c in independent of k. We present the proof for $1 < p < \infty$. The proof for $p = \infty$ is analogous, with the usual modifications. We can easily see that

$$\sigma_{\varepsilon,k}\|\widetilde{u}|L_p(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})\| \leq c_4^{\frac{1}{p}} h(2^{-\varepsilon k})^{-\frac{1}{p}}, \tag{4.2.17}$$

where c_4 is as in (4.2.10). Let $\delta \in (0, 1-\overline{s}(\sigma))$. Then

$$\begin{aligned}
&\int_{\Gamma_{\varepsilon k}}\int_{\Gamma_{\varepsilon k}} \frac{\Lambda(2^{\varepsilon k}|t-v|^{-1})^p}{h^*_{\varepsilon,k}(|t-v|)} |\widetilde{u}(t)-\widetilde{u}(v)|^p d\mu_{\varepsilon k}(t) d\mu_{\varepsilon k}(v) \\
&\quad\le \beta_{\varepsilon,k}^{-p}\, h(2^{-\varepsilon k})^{-1} \int_{\Gamma_{\varepsilon k}}\int_{\Gamma_{\varepsilon k}} \frac{\Lambda(2^{\varepsilon k}|t-v|^{-1})^p}{h^*_{\varepsilon,k}(|t-v|)} |t-v|^p d\mu_{\varepsilon k}(t) d\mu_{\varepsilon k}(v) \\
&\quad\lesssim \beta_{\varepsilon,k}^{-p}\, h(2^{-\varepsilon k})^{-1} \int_{t\in\Gamma_{\varepsilon k}} \sum_{j=0}^{+\infty} \Lambda(2^{\varepsilon(k+j)})^p 2^{-\varepsilon j p} d\mu_{\varepsilon k}(t) \\
&\quad\lesssim 2^{\varepsilon\alpha k p} h(2^{-\varepsilon k})^{-1} \mu_{\varepsilon k}(\Gamma_{\varepsilon k}) \sum_{j=0}^{+\infty} \Big(\frac{\sigma_{\varepsilon,k+j}}{\sigma_{\varepsilon,k}}\Big)^p 2^{-\varepsilon j p} \\
&\quad\lesssim 2^{\varepsilon\alpha k p} h(2^{-\varepsilon k})^{-1} \sum_{j=0}^{+\infty} 2^{-\varepsilon j p(1-(\overline{s}(\sigma)+\delta))} \\
&\quad\lesssim 2^{\varepsilon\alpha k p} h(2^{-\varepsilon k})^{-1}
\end{aligned}$$

Then $\widetilde{u} \in \widetilde{B}_p^{T_k(\sigma_\varepsilon),N_\varepsilon}(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})$. By Theorems 4.31 and 3.16 and (4.2.17), we can conclude that $\mathscr{E}_{\varepsilon,k}\widetilde{u} \in B_p^{T_k(a_\varepsilon),N_\varepsilon}(\mathbb{R}^n)$ and

$$a = \sigma \boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}} = (\alpha)\beta \boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}},$$

and

$$\|\mathscr{E}_{\varepsilon,k}\widetilde{u} | B_p^{T_k(a_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\| \lesssim 2^{\varepsilon\alpha k} 2^{\frac{\varepsilon k n}{p}}.$$

We consider a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\varphi(x) = 1 \quad \text{if} \quad |x| \le d \qquad \text{and} \qquad \varphi(x) = 0 \quad \text{if} \quad |x| \ge 2d.$$

Recalling that $\widetilde{u}$ denotes the function $a_\Gamma^{k,m}(2^{-\varepsilon k}\cdot)$, let

$$a^{k,m} := \varphi\big(2^{\varepsilon k}(\cdot - \gamma^{k,m})\big) \cdot \mathscr{E}_{\varepsilon,k}\big(a_\Gamma^{k,m}(2^{-\varepsilon k}\cdot)\big).$$

Then

$$\|a^{k,m} | B_p^{T_k(a_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\| \lesssim 2^{\varepsilon\alpha k} 2^{\frac{\varepsilon k n}{p}}, \qquad \operatorname{supp} a^{k,m} \subset B(2^{\varepsilon k}\gamma^{k,m}, 2d) \tag{4.2.18}$$

and

$$\operatorname{tr}_{\mathbf{F}_{\varepsilon k}}(a_\Gamma^{k,m}(2^{-\varepsilon k}\cdot)) = a_\Gamma^{k,m}(2^{-\varepsilon k}\cdot). \tag{4.2.19}$$

We refer to Remark 4.33 for some comments about (4.2.19).

Let

$$b^{k,m} := a^{k,m}(2^{\varepsilon k}\cdot).$$

By Remark 3.19, $b^{k,m} \in B_p^{a_\varepsilon,N_\varepsilon}(\mathbb{R}^n)$ and so, applying Proposition 3.4 and Theorem 3.18, we obtain

$$\|b^{k,m} | B_p^a(\mathbb{R}^n)\| \sim \|b^{k,m} | B_p^{a_\varepsilon,N_\varepsilon}(\mathbb{R}^n)\| \sim 2^{-\frac{\varepsilon k n}{p}} \|a^{k,m} | B_p^{T_k(a_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\| \lesssim 2^{\varepsilon\alpha k},$$

where we also used (4.2.18). As $\operatorname{supp} b^{k,m} \subset B(\gamma^{k,m}, 2d2^{-\varepsilon k})$ and $a = (\alpha)\beta\boldsymbol{h}^{1/p}(n)^{1/p}$, then, up to the multiplication by convenient constants, $b^{k,m}$ are $2d$-$(\beta\boldsymbol{h}^{1/p}(n)^{1/p}, p)_\alpha$-$\varepsilon$-

atoms according to Definition 3.21. Let

$$g = \sum_{k=0}^{\infty} \sum_{m=1}^{M_k} \nu_{k,m} b^{k,m}(x) \qquad \text{in } L_p(\mathbb{R}^n).$$

By Theorem 3.23,

$$g \in B_p^{\beta \boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}(\mathbb{R}^n) \quad \text{and} \quad \|g|B_p^{\beta \boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}(\mathbb{R}^n)\| \lesssim \|\nu|b_p^{\Gamma}\|.$$

Let us prove that $\operatorname{tr}_\Gamma g = f$. Let $T \in \mathbb{N}_0$. Then

$$\begin{aligned} \|f - \operatorname{tr}_\Gamma g|L_p(\Gamma)\| \le & \left\| f - \sum_{k=0}^{T} \sum_{m=1}^{M_k} \nu_{k,m} a_\Gamma^{k,m} |L_p(\Gamma)\right\| \\ & + \left\| \sum_{k=0}^{T} \sum_{m=1}^{M_k} \nu_{k,m} a_\Gamma^{k,m} - \operatorname{tr}_\Gamma \sum_{k=0}^{\infty} \sum_{m=1}^{M_k} \nu_{k,m} b^{k,m} |L_p(\Gamma)\right\|. \end{aligned} \tag{4.2.20}$$

By (4.2.16), the first expression in (4.2.20) converges to 0 when $T \to \infty$. The second can be estimated as follows

$$\begin{aligned} \left\| \operatorname{tr}_\Gamma \sum_{k=T+1}^{\infty} \sum_{m=1}^{M_k} \nu_{k,m} b^{k,m} |L_p(\Gamma)\right\| & \lesssim \left\| \sum_{k=T+1}^{\infty} \sum_{m=1}^{M_k} \nu_{k,m} b^{k,m} |B_p^{\beta \boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}(\mathbb{R}^n)\right\| \\ & \lesssim \left(\sum_{k=T+1}^{\infty} \sum_{m=1}^{M_k} |\nu_{k,m}|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Letting $T \to \infty$ we conclude the proof. ∎

Remark 4.35. Following the same procedure in Remark 3.24 we can prove that in the conditions of Theorem 4.34, if $\nu \in b_p^\Gamma$, then

$$\sum_{k=0}^{\infty} \sum_{m=1}^{M_k} \nu_{k,m} a_\Gamma^{k,m}(\gamma),$$

converges in $L_p(\Gamma)$, for all families of d-$(\beta_\varepsilon, p)_\Gamma^*$-$\varepsilon$-atoms [respectively for d-$(\beta, p, \varepsilon)_\Gamma^{**}$-atoms] $a_\Gamma^{k,m}$, $k \in \mathbb{N}_0$, $m \in \{1, ..., M_k\}$.

5. Besov spaces on quasi-metric spaces

5.1. Quasi-metric spaces and Euclidean charts. In this subsection we present some basic assertions about quasi-metric spaces and also the concept of *Euclidean charts*. We recall once more that we follow the approach in [28].

Definition 5.1. Let X be a (non-empty) set. A function $\varrho : X \times X \to [0, \infty)$ is a *quasi-metric* if

$$\varrho(x, y) = 0 \text{ if, and only if, } x = y,$$

$$\varrho(x, y) = \varrho(y, x) \text{ for all } x, y \in X,$$

there is a number $A \geq 1$ such that for all $x, y, z \in X$

$$\varrho(x,y) \leq A[\varrho(x,z) + \varrho(z,y)]. \tag{5.1.1}$$

If (5.1.1) is true with $A = 1$ then ϱ is a *metric.*

In what follows we will use the following notation

$$B^X(x,r) := \{y \in X : \varrho(x,y) < r\}, \qquad x \in X, \quad r > 0.$$

Useful properties about quasi-metrics spaces are given in the next theorem.

Theorem 5.2. *Let ϱ be a quasi-metric on a set X.*
(i) There is a number ε_0 with $0 < \varepsilon_0 \leq 1$ and a quasi-metric $\overline{\varrho}$ such that $\varrho \sim \overline{\varrho}$ and, for any $0 < \varepsilon \leq \varepsilon_0$, $\overline{\varrho}^\varepsilon$ is a metric.
(ii) Let $0 < \varepsilon \leq \varepsilon_0$. There is a positive number c such that for all $x \in X$, $y \in X$, $z \in X$,

$$|\overline{\varrho}(x,y) - \overline{\varrho}(x,z)| \leq c\,\overline{\varrho}(x,y)^\varepsilon\,[\overline{\varrho}(x,y) + \overline{\varrho}(x,z)]^{1-\varepsilon}. \tag{5.1.2}$$

Remark 5.3. For proofs of part (i) we refer to [13, p. 110-112, Proposition 14.5] and [28, p. 25, Remark 3.2]. In the latter it was also remarked that, though (5.1.2) is known since some time (cf. [18, p. 259, Theorem 2]), it can also be obtained as a consequence of (i). This property plays an important role in the analysis on quasi-metric spaces. It paves the way to introduce a topology on X taking the balls

$$B^X_{\overline{\varrho}}(x,r) = \{y \in X : \overline{\varrho}(x,y) < r\}, \quad r > 0,$$

as a basis of neighborhoods of $x \in X$.

Definition 5.4. Let ϱ be a quasi-metric on a set X equipped with the topology as just indicated.
(i) (X, ϱ, μ) is called a *space of homogeneous type* if μ is a non-negative regular Borel measure on X such that there is a constant A' with

$$0 < \mu(B^X(x,2r)) \leq A'\,\mu(B^X(x,r)), \quad \text{for all } x \in X,\ r > 0, \tag{5.1.3}$$

(doubling condition).

(ii) Let $h \in \mathbb{H}$ according to Definition 4.1. Then (X, ϱ, μ) is called an *h-space* if it is a complete space of homogeneous type as in part (i) with

$$\operatorname{Diam} X = \sup\{\varrho(x,y) : x, y \in X\} < \infty \tag{5.1.4}$$

and

$$\mu(B^X(x,r)) \sim h(r) \quad \text{for all } x \in X \text{ and } 0 < r \leq \operatorname{Diam} X$$

Remark 5.5. In the last decade, a lot has been done to develop a substancial analysis on spaces of homogeneous type. In [29, 1.17.4, 8.2] several references are given.
The above notation for d-spaces was introduced in [30] imitating the notation of d-sets in $\mathbb{R}^n$. We refer to [12] where Besov and Triebel-Lizorkin spaces with classical smoothness on d-spaces were studied in detail. In [31], Yang proved that, under some restrictions, Besov spaces defined on a d-set regarded as a d-space coincide with Besov spaces defined on d-sets using Triebel's methods, based on traces and quarkonial decompositions (cf. [25] and [26]).

The next theorem paves the way to relate quasi-metric spaces and fractal sets in some $\mathbb{R}^n$.

THEOREM 5.6. *Let* (X, ϱ, μ) *be a space of homogeneous type. Let* $0 < \varepsilon_0 \leq 1$ *be the number mentioned in Theorem 5.2 and let* $0 < \varepsilon < \varepsilon_0$. *Then there is an* $n \in \mathbb{N}$ *and a bi-Lipschitzian mapping*

$$L : X \to \mathbb{R}^n$$

from $(X, \varrho^\varepsilon, \mu)$ *into* $\mathbb{R}^n$. *This means that*

$$\varrho^\varepsilon(x, y) \sim |L(x) - L(y)|, \quad x, y \in X. \tag{5.1.5}$$

The dimension n *and the bi-Lipschitzian constants in* (5.1.5) *can be chosen to depend only on* ε *and on the doubling constant* A' *in* (5.1.3).

REMARK 5.7. This theorem was firstly proved by P. Assouad (cf. [1, 2]). More information about this may be found in [13].
If ϱ is a quasi-metric and $\varepsilon < \varepsilon_0$, for ε_0 as in Theorem 5.2, we say that $(X, \varrho^\varepsilon, \mu)$ is a snowflaked version of (X, ϱ, μ). In particular, if ϱ is a metric, a snowflaked version of (X, ϱ, μ) is a structure $(X, \varrho^\varepsilon, \mu)$ with $\varepsilon < 1$.

PROPOSITION 5.8. *Let* (X, ϱ, μ) *be an* h*-space and let* $\varepsilon_0, \varepsilon$ *and* L *be as in Theorem 5.6. Let* $h_{1/\varepsilon}$ *be given by*

$$h_{1/\varepsilon}(r) := h(r^{1/\varepsilon}), \quad r > 0. \tag{5.1.6}$$

Then $\Gamma = L(X) \subset \mathbb{R}^n$ *is an* $h_{1/\varepsilon}$*-set.*

Proof. For $r > 0$ we denote by $B^X_\varepsilon(x, r)$ the balls

$$B^X_\varepsilon(x, r) = \{y \in X : \varrho^\varepsilon(x, y) < r\}.$$

Hence, for all $x \in X$ and $r > 0$,

$$B^X_\varepsilon(x, r) = B^X(x, r^{1/\varepsilon}).$$

As L is a bi-Lipschitzian mapping from $(X, \varrho^\varepsilon, \mu)$ into $\mathbb{R}^n$, there are $0 < a_1 \leq a_2$ with

$$a_1 \varrho^\varepsilon(x, y) \leq |L(x) - L(y)| \leq a_2 \varrho^\varepsilon(x, y), \quad \text{for all } x, y \in X. \tag{5.1.7}$$

By (5.1.7), (5.1.4) and the assumption that X is complete it follows that $\Gamma = L(X)$ is compact.
Considering $\nu = \mu \circ L^{-1}$, by (5.1.7) we obtain

$$\nu(B(\gamma, r)) \sim h_{1/\varepsilon}(r), \quad \gamma \in \Gamma,\ 0 < r \leq \operatorname{Diam} \Gamma.$$

■

REMARK 5.9. So, in the previous conditions, $h_{1/\varepsilon}$ is a measure function in $\mathbb{R}^n$ and $\nu \sim \mathcal{H}^{h_{1/\varepsilon}}_\Gamma$, where $\mathcal{H}^{h_{1/\varepsilon}}_\Gamma$ is the restriction of the Hausdorff measure $\mathcal{H}^{h_{1/\varepsilon}}_\Gamma$ in $\mathbb{R}^n$ to Γ.

DEFINITION 5.10. Let (X, ϱ, μ) be an h-space and let $0 < \varepsilon < \varepsilon_0$ where ε_0 is the same number as in Theorem 5.2. We say that $(X, \varrho, \mu; L)$ or, for short, $(X; L)$ is an *Euclidean* ε*-chart* or an ε*-chart* if L is a bi-Lipschitzian map from $(X, \varrho^\varepsilon, \mu)$ onto $(\Gamma, \varrho_n, \mathcal{H}^{h_{1/\varepsilon}}_\Gamma)$, where $h_{1/\varepsilon}$ is as in (5.1.6) and ϱ_n denotes the usual metric in $\mathbb{R}^n$.

5.2. Function spaces on h-spaces.

DEFINITION 5.11. Let (X, ϱ, μ) be an h-space with an ε-chart $(X; L)$. Consider $\nu = \mu \circ L^{-1}$, $h_{1/\varepsilon}$ the function in (5.1.6) and $\Gamma = L(X) \subset \mathbb{R}^n$, where n is chosen large enough so that

$$-\underline{s}(\boldsymbol{h}_{1/\varepsilon}) = -\frac{1}{\varepsilon}\,\underline{s}(\boldsymbol{h}) < n,$$

i.e., so that Γ satisfies the ball condition (we refer to Corollary 4.8). Let σ be an admissible sequence such that $\underline{s}(\sigma) > 0$ and consider $\sigma_{1/\varepsilon}$ according to the notation introduced in (2.2.3). Let $0 < p, q \leq \infty$. Then

$$B^{\sigma}_{p,q}(X, \varrho, \mu; L) := \mathbb{B}^{\sigma_{1/\varepsilon}}_{p,q}(\Gamma, \varrho_n, \nu) \circ L, \tag{5.2.1}$$

i.e., $f \in B^{\sigma}_{p,q}(X, \varrho, \mu; L)$ if and only if $f = g \circ L$ for some $g \in \mathbb{B}^{\sigma_{1/\varepsilon}}_{p,q}(\Gamma, \varrho_n, \nu)$ and

$$\|f \,|\, B^{\sigma}_{p,q}(X, \varrho, \mu; L)\| := \|g \,|\, \mathbb{B}^{\sigma_{1/\varepsilon}}_{p,q}(\Gamma, \varrho_n, \nu)\|.$$

REMARK 5.12. We schematise the construction described:

$$\begin{array}{ccccc} \underset{h\text{-space}}{(X, \varrho, \mu)} & \underset{\text{snowfl.}}{\rightsquigarrow} & (X, \varrho^{\varepsilon}, \mu) & \underset{L}{\rightarrow} & \underset{h_{1/\varepsilon}\text{-set}}{(\Gamma, \varrho_n, \nu).} \\ B^{\sigma}_{p,q}(X; L) & & & & \mathbb{B}^{\sigma_{1/\varepsilon}}_{p,q}(\Gamma) \end{array}$$

The space $B^{\sigma}_{p,q}(X, \varrho, \mu; L)$, or just $B^{\sigma}_{p,q}(X; L)$, is a quasi-Banach space (Banach space if $p \geq 1$ and $q \geq 1$).

If (X, ϱ, μ) is an h-set (Γ, ϱ_n, ν) in some $\mathbb{R}^n$ and if L is the identity, then it follows immediately from Definition 5.11 that $B^{\sigma}_{p,q}(X; L) = \mathbb{B}^{\sigma}_{p,q}(\Gamma)$. But, if we take different functions L we may introduce different scales of spaces in $(\Gamma, \varrho_n, \nu; L)$ which might possibly not be obtained from trace spaces according to Definition 4.11. Nevertheless, we will prove that, under some conditions with respect to h, σ and p, for a certain range of values of ε, the spaces $B^{\sigma}_{p}(X; L)$ do not depend on the ε-chart considered.

REMARK 5.13. As we have already mentioned, the use of Euclidean charts to define Besov spaces on quasi-metric spaces was introduced by Triebel in [28]. In that paper the spaces $B^{(s)}_{p}(X, \varrho, \mu; L)$, where $s > 0$, $1 < p < \infty$ and (X, ϱ, μ) is a d-space, were defined using a kind of quarkonial decompositions (cf. [28, p. 34, Definition 4.6]). The adapted quarks for $B^{(s)}_{p}(X; L)$ are the composition of the quarks for the corresponding space $\mathbb{B}^{(s/\varepsilon)}_{p}(\Gamma)$ with the ε-chart L.

It is an immediate consequence of Definition 5.11 and the existence, under some restrictions, of characterisations with quarkonial decompositions for spaces $\mathbb{B}^{\sigma_{1/\varepsilon}}_{p,q}(\Gamma)$ (cf. [3] and [17]) that something analogous could be obtained for the spaces defined above.

THEOREM 5.14. *Let $1 < p \leq \infty$ and σ be an admissible sequence. For $i \in \{1, 2\}$, let $(X, \varrho, \mu; L_i)$ be ε_i-charts of an h-space (X, ϱ, μ). If*

$$\overline{s}(\boldsymbol{h}) < 0 \quad \text{and} \quad 0 < \underline{s}(\sigma) \leq \overline{s}(\sigma) < \min\{\varepsilon_1, \varepsilon_2\}, \tag{5.2.2}$$

then

$$B^{\sigma}_{p}(X; L_1) = B^{\sigma}_{p}(X; L_2). \tag{5.2.3}$$

Proof. We recall that, for $i = 1, 2$,

$$\underset{h\text{-space}}{(X,\varrho,\mu)} \underset{\text{snowfl.}}{\rightsquigarrow} (X,\varrho^{\varepsilon_i},\mu) \underset{L_i}{\longrightarrow} \underset{h_{1/\varepsilon_i}\text{-set}}{(\Gamma_i,\varrho_{n_i},\nu_i)}\quad ,$$
$$B_p^\sigma(X;L_i) \qquad\qquad \mathbb{B}_p^{\sigma_{1/\varepsilon_i}}(\Gamma_i,\varrho_{n_i},\nu_i)$$

where

$$\Gamma_i = L_i(X) \subset \mathbb{R}^{n_i}, \qquad \nu_i = \mu \circ L_i^{-1}$$

and ϱ_{n_i} denotes the usual metric in $\mathbb{R}^{n_i}$. As previously, n_i, $i = 1, 2$, are chosen conveniently large such that

$$-\frac{1}{\varepsilon_i}\,\underline{s}(\boldsymbol{h}) < n_i.$$

By (5.2.1), (5.2.3) is equivalent to

$$\mathbb{B}_p^{\sigma_{1/\varepsilon_1}}(\Gamma_1,\varrho_{n_1},\nu_1) = \mathbb{B}_p^{\sigma_{1/\varepsilon_2}}(\Gamma_2,\varrho_{n_2},\nu_2) \circ L_2 \circ L_1^{-1}. \tag{5.2.4}$$

Let us prove that, given $g_1 \in \mathbb{B}_p^{\sigma_{1/\varepsilon_1}}(\Gamma_1,\varrho_{n_1},\nu_1)$, there is $g_2 \in \mathbb{B}_p^{\sigma_{1/\varepsilon_2}}(\Gamma_2,\varrho_{n_2},\nu_2)$ such that

$$g_1 = g_2 \circ L_2 \circ L_1^{-1} \quad \text{in} \quad \Gamma_1$$

and

$$\|g_2|\mathbb{B}_p^{\sigma_{1/\varepsilon_2}}(\Gamma_2,\varrho_{n_2},\nu_2)\| \lesssim \|g_1|\mathbb{B}_p^{\sigma_{1/\varepsilon_1}}(\Gamma_1,\varrho_{n_1},\nu_1)\|. \tag{5.2.5}$$

For all $\gamma_1, \delta_1 \in \Gamma_1$,

$$\varrho_{n_2}(L_2 \circ L_1^{-1}(\gamma_1), L_2 \circ L_1^{-1}(\delta_1)) \sim \varrho_{n_1}^{\varepsilon_2/\varepsilon_1}(\gamma_1,\delta_1). \tag{5.2.6}$$

We assume that $\varepsilon_2 \le \varepsilon_1$ and fix, for all $j \in \mathbb{N}_0$,

$$\{\gamma^{j,m} : m = 1, ..., M_j\} \subset \Gamma_2$$

$2^{-\varepsilon_2/\varepsilon_1}$-approximate lattices for Γ_2. It follows from Remark 4.15 that

$$M_j \sim h_{1/\varepsilon_2}(2^{-\frac{\varepsilon_2}{\varepsilon_1}j}) = h(2^{-\frac{j}{\varepsilon_1}}), \quad j \in \mathbb{N}_0.$$

By (5.2.6), for all $j \in \mathbb{N}_0$,

$$\{\delta^{j,m} : m = 1, ..., M_j\} \quad \text{with} \quad \delta^{j,m} = (L_1 \circ L_2^{-1})(\gamma^{j,m}),$$

are 2^{-j}-approximate lattices for Γ_1.

Let $g_1 \in \mathbb{B}_p^{\sigma_{1/\varepsilon_1}}(\Gamma_1,\varrho_{n_1},\nu_1)$. By Proposition 2.12 and (5.2.2)

$$0 < \underline{s}(\sigma_{1/\varepsilon_1}) \le \overline{s}(\sigma_{1/\varepsilon_1}) < \frac{\varepsilon_2}{\varepsilon_1}.$$

Hence, applying Theorem 4.34, there is $\lambda \in b_p^{\Gamma_1}$ such that

$$g_1 = \sum_{j=0}^{\infty}\sum_{m=1}^{M_j} \lambda_{j,m} a_{\Gamma_1}^{j,m} \quad \text{in } L_p(\Gamma_1,\nu_1),$$

where, for $j \in \mathbb{N}_0$ and $m \in \{1, ..., M_j\}$, $a_{\Gamma_1}^{j,m}$ are $\varepsilon_2/\varepsilon_1$-$d$-$(\sigma_{1/\varepsilon_1},p)_{\Gamma_1}^{**}$-atoms located in $B(\delta^{j,m}, d2^{-j})$, for a conveniently chosen d, and

$$\|\lambda|b_p^{\Gamma_1}\| \lesssim \|g_1|\mathbb{B}_p^{\sigma_{1/\varepsilon_1}}(\Gamma_1,\varrho_{n_1},\nu_1)\|.$$

Let

$$a^{j,m}_{\Gamma_2} := a^{j,m}_{\Gamma_1} \circ L_1 \circ L_2^{-1}, \quad j \in \mathbb{N}_0, \quad m \in \{1, ..., M_j\}.$$

The functions $a^{j,m}_{\Gamma_2}$ are $\varepsilon_2/\varepsilon_1$-$d'$-$(\sigma_{1/\varepsilon_2}, p)^*_{\Gamma_2}$-atoms located at $B(\gamma^{j,m}, d' 2^{-\varepsilon_2/\varepsilon_1 j})$, for some conveniently chosen d'. Then, according to Remark 4.35,

$$\sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda_{j,m} a^{j,m}_{\Gamma_2}$$

converges in $L_p(\Gamma_2, \nu_2)$ to, say, g_2. Again by Theorem 4.34, we conclude that $g_2 \in \mathbb{B}^{\sigma_{1/\varepsilon_2}}_p(\Gamma_2, \varrho_{n_2}, \nu_2)$ and

$$\|g_2 | \mathbb{B}^{\sigma_{1/\varepsilon_2}}_p(\Gamma_2, \varrho_{n_2}, \nu_2)\| \lesssim \|\lambda | b^{\Gamma_1}_p\|.$$

Hence $g_2 = g_1 \circ L_1 \circ L_2^{-1}$ and (5.2.5).
The reverse inclusion in (5.2.4) is proved analogously. ■

REMARK 5.15. We can interprete $L_2 \circ L_1^{-1}$ as being an $\varepsilon_2/\varepsilon_1$-chart of the h_{1/ε_1}-space $(\Gamma_1, \varrho_{n_1}, \nu_1)$. Hence, (5.2.4) can be written as

$$B^{\sigma_{1/\varepsilon_1}}_p(\Gamma_1; L_2 \circ L_1^{-1}) = \mathbb{B}^{\sigma_{1/\varepsilon_1}}_p(\Gamma_1),$$

based on

$$\begin{array}{ccccc} \underset{h_{1/\varepsilon_1}\text{-space}}{(\Gamma_1, \varrho_{n_1}, \nu_1)} & \underset{\text{snowfl.}}{\rightsquigarrow} & (\Gamma_1, \varrho^{\varepsilon_2/\varepsilon_1}_{n_1}, \nu_1) & \underset{L_2 \circ L_1^{-1}}{\longrightarrow} & \underset{h_{1/\varepsilon_2}\text{-set}}{(\Gamma_2, \varrho_{n_2}, \nu_2).} \\ B^{\sigma_{1/\varepsilon_1}}_p(\Gamma_1; L_2 \circ L_1^{-1}) & & & & \mathbb{B}^{\sigma_{1/\varepsilon_2}}_p(\Gamma_2) \end{array}$$

So, to prove that the definition of Besov spaces on abstract h-spaces using ε-charts is independent of the charts, under some restrictions, it was enough to prove that this construction works (in the sense of being independent of the charts) in the particular case of h-sets.

The next definition corresponds to the abstract version of Definition 4.14.

DEFINITION 5.16. Let (X, ϱ, μ) be an h-space and $j \in \mathbb{N}_0$. We say that

$$\{x^{j,m}\}_{m=1}^{M_j} \subset X$$

with $j \in \mathbb{N}_0$, is a $2^{-j}-$*approximate lattices for* X if there exist positive numbers c_1 and c_2 independent of j such that

$$\varrho(x^{j,m_1}, x^{j,m_2}) \geq c_1 2^{-j}, \quad j \in \mathbb{N}_0, \quad m_1 \neq m_2,$$

$$X = \bigcup_{m=1}^{M_j} B_{j,m} \quad \text{with} \quad B_{j,m} = B^X(x^{j,m}, c_2 2^{-j}) \quad \text{for } j \in \mathbb{N}_0. \tag{5.2.7}$$

REMARK 5.17. Let us note that if we consider an Euclidean ε-chart of (X, ϱ, μ), L, with $L(X) = \Gamma$, then

$$\gamma_1, \gamma_2 \in \Gamma \quad \Leftrightarrow \quad \gamma_1 = L(x_1) \quad \text{and} \quad \gamma_2 = L(x_2), \quad x_1, x_2 \in X$$

and so

$$|\gamma_1 - \gamma_2| \sim 2^{-\varepsilon j} \quad \Leftrightarrow \quad \varrho(x_1, x_2) \sim 2^{-j}$$

As a result, the existence of 2^{-j}-approximate lattices for X follows from the existence of ε-charts and $2^{-\varepsilon j}$-approximate lattices for corresponding sets in $\mathbb{R}^n$ as in Definition 4.14. If, for $j \in \mathbb{N}_0$,

$$\{\gamma^{j,m} : m = 1, ..., M_j\} \subset \Gamma$$

is a $2^{-\varepsilon j}$-approximate lattice for the $h_{1/\varepsilon}$-set Γ, then

$$\{x^{j,m} : m = 1, ..., M_j\} \subset X \qquad \text{with} \quad x^{j,m} = L^{-1}(\gamma^{j,m})$$

is a 2^{-j}-approximate lattice for X. Moreover, according to Remark 4.15, for all $j \in \mathbb{N}_0$,

$$M_j \sim h_{1/\varepsilon}(2^{-\varepsilon j})^{-1} = h(2^{-j})^{-1}.$$

Definition 5.18. Let $h \in \mathbb{H}$, (X, ϱ, μ) be an h-space and $\varepsilon \in (0, \varepsilon_0)$, for ε_0 as in Theorem 5.2. Let $\{x_{j,m}\}_{m=1}^{M_j}$, $j \in \mathbb{N}_0$, be 2^{-j}-approximate lattices for X and $d > c_2$, where c_2 is as in (5.2.7). Consider an admissible sequence σ and $1 < p \leq \infty$. A function on X, $a_X^{j,m}$, is called an d-$(\sigma, p, \varepsilon)_X$-atom if for $j \in \mathbb{N}_0$ and $m = 1, ..., M_j$,

(a) $\operatorname{supp} a_X^{j,m} \subset B(x^{j,m}, d2^{-j})$

(b) $|a_X^{j,m}(x)| \leq \sigma_j^{-1}\, h(2^{-j})^{-\frac{1}{p}}, \qquad x \in X,$

(c) $|a_X^{j,m}(x) - a_X^{j,m}(y)| \leq \sigma_j^{-1}\, h(2^{-j})^{-\frac{1}{p}} 2^{\varepsilon j} \varrho^{\varepsilon}(x, y), \qquad x, y \in X.$

Theorem 5.19. *Let $(X, \varrho, \mu; L)$ be an ε-chart of an h-space (X, ϱ, μ). Let $1 < p \leq \infty$ and σ be an admissible sequence. Assume that*

$$\overline{s}(\boldsymbol{h}) < 0 \quad \textit{and} \quad 0 < \underline{s}(\sigma) \leq \overline{s}(\sigma) < \varepsilon. \tag{5.2.8}$$

Let $d > c_2$, where c_2 is as in (5.2.7). *Then $B_p^{\sigma}(X; L)$ is the collection of all $f \in L_p(X)$ which can be represented as*

$$f = \sum_{j=1}^{\infty} \sum_{m=1}^{M_j} \lambda_{j,m} a_X^{j,m}(x), \qquad \textit{in} \quad L_p(X), \tag{5.2.9}$$

for some $\lambda \in b_p^{L(X)}$, where $a_X^{j,m}$ are d-$(\sigma, p, \varepsilon)_X$-atoms according to Definition 5.18. Furthermore,

$$\|f|B_p^{\sigma}(X; L)\| \sim \inf \|\lambda|b_p^{L(X)}\|,$$

where the infimum is taken over all representations (5.2.9).

Proof. One can easily see that $a_X^{j,m}$ is a d-$(\sigma, p, \varepsilon)_X$-atom if, and only if, $a_X^{j,m} \circ L^{-1}$ is a d'-$(\sigma_{1/\varepsilon}, p)_\Gamma^*$-$\varepsilon$-atom, where $\Gamma = L(X)$ is an $h_{1/\varepsilon}$-set and d' is conveniently chosen. Hence, the above result can be obtained using ε-charts and applying Theorem 4.34 to the spaces $\mathbb{B}_p^{\sigma_{1/\varepsilon}}(\Gamma)$. ■

Remark 5.20. If $h(r) = r^d$ and $\sigma = (s)$, then (5.2.8) corresponds to assume $d > 0$ and

$$0 < s < \varepsilon,$$

which coincides with the conditions obtained by Triebel in [28, p. 42, Theorem 4.22]. In this work, to guarantee the uniqueness of the spaces $B_p^{(s)}(X; L)$ (where X is a d-space), instead of a direct proof as it was done in Theorem 5.14, transferring everything to function spaces on special sets in $\mathbb{R}^n$, Triebel used a result corresponding to the above one to conclude that the spaces $B_p^{(s)}(X;L)$ coincide with the spaces considered by Han and Yang in [12] and, consequently, are independent of the Euclidean charts.

5.3. Example: entropy numbers. In this subsection we will present an example which shows that this approach for the definition of the function spaces allows to develop a theory for function spaces on quasi-metric spaces, taking advantage of what is already known for function spaces on fractals in $\mathbb{R}^n$.

DEFINITION 5.21. Let A and B be two quasi-Banach spaces and let $T : A \to B$ be a linear and bounded operator. Then for all $j \in \mathbb{N}$, the j-th (dyadic) *entropy number* of T is defined by

$$e_j(T) = \inf\{\delta > 0 : T(\mathcal{B}_A) \subset \bigcup_{l=1}^{2^{j-1}} (b_l + \delta\mathcal{B}_{\mathcal{B}}), \text{ for some } b_1, ..., b_{2^{j-1}} \in B\},$$

where $\mathcal{B}_A$ and $\mathcal{B}_B$ denote the closed unitary balls in A and in B, respectively.

REMARK 5.22. If $(\alpha_j)_{j\in\mathbb{N}}$ is an increasing sequence of positive numbers we write e_{α_j} instead of $e_{[\alpha_j]}$, where $[\cdot]$ denotes the integer-part function.

In the next Proposition we present estimates for the entropy numbers of embeddings between function spaces on h-spaces.

PROPOSITION 5.23. *Let $h \in \mathbb{H}$ be such that $\overline{s}(\boldsymbol{h}) < 0$. Consider an h-space (X, ϱ, μ) with an ε-chart $(X;L)$, according to Definition 5.10.*
Let σ and τ be admissible sequences such that $0 < \underline{s}(\tau) < \underline{s}(\sigma)$. Consider $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \le \infty$ and

$$\underline{s}(\sigma\tau^{-1}) > -\underline{s}(\boldsymbol{h})\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+.$$

Then the embedding

$$id^X : B^{\sigma}_{p_1,q_1}(X;L) \hookrightarrow B^{\tau}_{p_2,q_2}(X;L)$$

is compact and

$$e_{h_j^{-1}}(id^X) \sim \tau_j\sigma_j^{-1}. \tag{5.3.1}$$

Proof. Using ε-charts this result is just a consequence of a corresponding one for Besov spaces on h-sets, which was proved by Bricchi (cf. [3, p. 130, Theorem 4.3.2]). In the application of this theorem it may be convenient to choose n sufficiently large, where n stands for the dimension of the Euclidean space which contains $L(X)$.
By (5.2.1)

$$\mathscr{L} : f \mapsto f \circ L \ : \ \mathbb{B}^{\sigma_{1/\varepsilon}}_{p,q}(\Gamma, \varrho_n, \nu) \ \hookrightarrow \ B^{\sigma}_{p,q}(X, \varrho, \mu; L)$$

is an isomorphic map.
We decompose id^X according to the following commutative diagram.

$$\begin{array}{ccc}
B^{\sigma}_{p_1,q_1}(X;L) & \overset{\mathscr{L}^{-1}}{\longrightarrow} & \mathbb{B}^{\sigma_{1/\varepsilon}}_{p_1,q_1}(\Gamma,\varrho_n,\nu) \\
id^X \Big\downarrow & & \Big\downarrow id^\Gamma \\
B^{\tau}_{p_2,q_2}(X;L) & \underset{\mathscr{L}}{\longleftarrow} & \mathbb{B}^{\tau_{1/\varepsilon}}_{p_2,q_2}(\Gamma,\varrho_n,\nu)
\end{array}$$

Hence,

$$e_{h_j^{-1}}(id^X) \sim e_{h_j^{-1}}(id^\Gamma).$$

By [3, p. 130, Theorem 4.3.2],

$$e_{h^{-1}_{1/\varepsilon,j}}(id^\Gamma) \sim \tau_{1/\varepsilon,j}\ \sigma^{-1}_{1/\varepsilon,j},$$

and, after some calculations, we obtain (5.3.1). ■

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