A homogeneity property for Besov spaces

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Abstract: A homogeneity property for some Besov spaces $B_{p,q}^s$ is proved. An analogous property for some $F_{p,q}^s$ spaces is already known.

1 Introduction

In the present note we prove the so-called *homogeneity property* for some Besov spaces on $\mathbb{R}^n$. We show that for $0 < p, q \le \infty$ and $s > \sigma_p$,

$$\|f(\lambda \cdot) \,|\, B_{p,q}^s(\mathbb{R}^n)\| \sim \lambda^{s-\frac{n}{p}} \|f \,|\, B_{p,q}^s(\mathbb{R}^n)\|, \tag{1}$$

for all $0 < \lambda \leq 1$ and all

$$f \in B_{p,q}^s(\mathbb{R}^n) \quad \text{with} \quad \operatorname{supp} f \subset \{x \in \mathbb{R}^n : |x| \leq \lambda\}. \qquad (2)$$

A corresponding property for the spaces $F_{p,q}^s(\mathbb{R}^n)$ may be found in [9, Corollary 5.16, p. 66]. It is of some use in connection with refined localization, non-smooth atoms, scaling properties and pointwise multipliers. We refer to [10, Section 4.2.2]. Here we prove this property for the spaces $B_{p,q}^s(\mathbb{R}^n)$, with $s > \sigma_p$. For the particular case where $\sigma_p < s < n/p$ this property can be obtained via atomic decompositions. This was the starting point for the main result presented in this note.

This note is organised as follows: firstly we give the necessary definitions; secondly we deal with equivalent quasi-norms for the elements of a certain subspaces of $B_{p,q}^s(\mathbb{R}^n)$; finally we present the main result referred above.

2 Preliminaries

First we introduce some standard notation and useful definitions. As usual, $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{R}^n$, $n \in \mathbb{N}$, stands for the n-dimensional real Euclidean space and $\mathbb{R} = \mathbb{R}^1$. We use the symbol "$\lesssim$" in

$$a_k \lesssim b_k \quad \text{or} \quad \phi(r) \lesssim \psi(r)$$

always to mean that there is a positive number c_1 such that

$$a_k \leq c_1 b_k \quad \text{or} \quad \phi(r) \leq c_1 \psi(r)$$

for all admitted values of the discrete variable k or the continuous variable x, where $(a_k)_k$, $(b_k)_k$ are non-negative sequences and ϕ, ψ are non-negative functions. We use the equivalence "$\sim$" in

$$a_k \sim b_k \quad \text{or} \quad \phi(r) \sim \psi(r)$$

for

$$a_k \lesssim b_k \quad \text{and} \quad b_k \lesssim a_k \qquad \text{or} \qquad \phi(r) \lesssim \psi(r) \quad \text{and} \quad \psi(r) \lesssim \phi(r).$$

As usual $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ equipped with the usual topology, and $\mathcal{S}'(\mathbb{R}^n)$ denotes its topological dual, the space of all tempered distributions on $\mathbb{R}^n$. Furthermore, $L_p(\mathbb{R}^n)$, with $0 < p \leq \infty$, stands for the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\|f|L_p(\mathbb{R}^n)\| := \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}}$$

with the appropriate modification if $p = \infty$.

We use the standard abbreviations

$$\bar{p} := \max(1, p) \quad \text{and} \quad \sigma_p := n\Big(\frac{1}{p} - 1\Big)_+ = n\Big(\frac{1}{\min\{1, p\}} - 1\Big). \qquad (3)$$

As usually, "domain" stands for "open set".

Next, we recall the definition of differences of functions. If f is an arbitrary complex-valued function on $\mathbb{R}^n$, $h \in \mathbb{R}^n$ and $M \in \mathbb{N}$, then

$$(\Delta_h^M f)(x) := \sum_{j=0}^{M} \binom{M}{j} (-1)^{M-j} f(x + jh), \quad x \in \mathbb{R}^n, \qquad (4)$$

where $\binom{M}{j}$ are the binomial coefficients.

Definition 1. Let

$$0 < p, q \leq \infty \quad \text{and} \quad \sigma_p < s < M \in \mathbb{N}. \qquad (5)$$

Then $B_{p,q}^s(\mathbb{R}^n)$ is the collection of all $f \in L_p(\mathbb{R}^n)$ such that

$$\|f|B_{p,q}^s(\mathbb{R}^n)\|_M := \|f|L_p(\mathbb{R}^n)\| + \bigg(\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M f \mid L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n} \bigg)^{1/q} \qquad (6)$$

(with the usual modification if $q = \infty$) is finite.

Remark 2. The above spaces are quasi-Banach spaces. They have a substantial history. The classical spaces, which means $1 \leq p, q \leq \infty$, $s > 0$, go back to S.M. Nikol'skij in the 1950s ($q = \infty$) and O.V. Besov around 1960. These spaces have been considered afterwards in great detail especially by the Russian school. This may be found in [5] and [1]. At least in this specification they are also denoted as Nikol'skij-Besov spaces. The extension to $s < 0$ (what is not of interest for our purpose) and to $p < 1$ (which is subject of the later considerations) was first done by J. Peetre at the end of the 1960s and the early 1970s in terms of Fourier-analytical definitions. We refer to [6]. Systematic studies of these spaces may be found in [7, 8]. The first chapters both of [8] and [10] are surveys (entitled *How to measure smoothness*) where one finds the history of these spaces. The spaces $B_{p,q}^s(\mathbb{R}^n)$ are nowadays often introduced in terms of Fourier-analytical decompositions. If p, q, s are restricted by (5) then these spaces coincide with the spaces as introduced in Definition 1. A proof may be found in [7, Section 2.5.12, Theorem, Remark 3, Corollary, pp. 110-114] covering also that

$$B_{\infty,\infty}^s(\mathbb{R}^n) = \mathcal{C}^s(\mathbb{R}^n), \quad s > 0,$$

are the usual Hölder-Zygmund spaces, characterized by

$$\sup_{x \in \mathbb{R}^n} |f(x)| + \sup_{x \in \mathbb{R}^n, 0 \neq h \in \mathbb{R}^n} |h|^{-s} |\Delta_h^M f(x)| < \infty$$

where again $s < M \in \mathbb{N}$. The quasi-norms in (6) are equivalent to each other for different values of M with $0 < s < M \in \mathbb{N}$. This justifies our omission of the subscript M in the sequel.

3 Equivalent quasi-metrics

We collect a few further notation.

Definition 3. Let Ω be a domain in $\mathbb{R}^n$. Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$.
(i) Then $L_p(\Omega)$ is the collection of all complex-valued Lebesgue measurable functions in Ω such that

$$\|f|L_p(\Omega)\| := \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}$$

(with the usual modification if $p = \infty$) is finite.
(ii) Then $B_{p,q}^s(\Omega)$ is the collection of all $f \in D'(\Omega)$ such that there is a $g \in B_{p,q}^s(\mathbb{R}^n)$ with $g|_\Omega = f$. Furthermore,

$$\|f|B_{p,q}^s(\Omega)\| := \inf \|g|B_{p,q}^s(\mathbb{R}^n)\|, \tag{7}$$

where the infimum is taken over all $g \in B_{p,q}^s(\mathbb{R}^n)$ such that its restriction $g|_\Omega$ to Ω coincides in $D'(\Omega)$ with f.
(iii) Then $\widetilde{B}_{p,q}^s(\overline{\Omega})$ is the closed subspace of $B_{p,q}^s(\mathbb{R}^n)$ given by

$$\widetilde{B}_{p,q}^s(\overline{\Omega}) := \{f \in B_{p,q}^s(\mathbb{R}^n) : \operatorname{supp} f \subset \overline{\Omega}\}. \tag{8}$$

Remark 4. These spaces are quasi-Banach spaces. Function spaces on domains have also been studied in detail from the very beginning of the theory of function spaces in 1950s and 1960s. We refer to the books in Remark 2 and to [7, Chapter 3],[8, Chapter 5] and [9, Section 5].

Proposition 5. *Let Ω be a bounded domain in $\mathbb{R}^n$ and*

$$0 < p \leq \infty, \quad 0 < q \leq \infty, \quad \sigma_p < s < M \in \mathbb{N}. \tag{9}$$

Then

$$\|f|B_{p,q}^s(\mathbb{R}^n)\| \sim \left(\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M f \mid L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n} \right)^{1/q}, \tag{10}$$

for all $f \in \widetilde{B}_{p,q}^s(\overline{\Omega})$ (equivalent quasi-norms), with the usual modification if $q = \infty$.

Proof. We consider
$$B_R = \{x \in \mathbb{R}^n : |x| < R\} \tag{11}$$
with $\overline{\Omega} \subset B_R$. Let $0 < q < \infty$. We recall that by Definition 1 we have (6) for the full space $B_{p,q}^s(\mathbb{R}^n)$. If $p < 1$, as $s > \sigma_p$, we get by well-known embedding theorems that $B_{p,q}^s(\mathbb{R}^n) \subset L_1(\mathbb{R}^n)$. Consequently, by (6), we obtain

$$\|f|L_1(\mathbb{R}^n)\| + \left(\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M f \mid L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n} \right)^{1/q} \lesssim \|f|B_{p,q}^s(\mathbb{R}^n)\|. \tag{12}$$

If $f \in \widetilde{B}_{p,q}^s(\overline{\Omega})$, then

$$\|f|L_p(\mathbb{R}^n)\| = \|f|L_p(B_R)\| \lesssim \|f|L_1(B_R)\| = \|f|L_1(\mathbb{R}^n)\|. \qquad (13)$$

Thus, using (6), we obtain, for $p < 1$,

$$\|f|B_{p,q}^s(\mathbb{R}^n)\| \lesssim \|f|L_1(\mathbb{R}^n)\| + \left(\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M f \mid L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n} \right)^{1/q}. \qquad (14)$$

Hence, for all $f \in \widetilde{B}_{p,q}^s(\overline{\Omega})$ and $0 < p \le \infty$,

$$\|f|B_{p,q}^s(\mathbb{R}^n)\| \sim \|f|L_{\overline{p}}(\mathbb{R}^n)\| + \left(\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M f \mid L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n} \right)^{1/q}. \qquad (15)$$

It remains to prove that there is a positive number c such that

$$\|f|L_{\overline{p}}(\mathbb{R}^n)\| \le c \left(\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M f \mid L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n} \right)^{1/q}, \qquad (16)$$

for all $f \in \widetilde{B}_{p,q}^s(\overline{\Omega})$. Assuming that there is no such c, then for every $j \in \mathbb{N}$ one finds a function $f_j \in \widetilde{B}_{p,q}^s(\overline{\Omega})$ which can be normalised such that

$$1 = \|f_j|L_{\overline{p}}(\mathbb{R}^n)\| > j \left(\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M f_j \mid L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n} \right)^{1/q}. \qquad (17)$$

Hence

$$
\begin{aligned}
\|f_j|_{B_R} \mid B_{p,q}^s(B_R)\| &\le \|f_j|B_{p,q}^s(\mathbb{R}^n)\| \\
&\sim \|f_j|L_{\overline{p}}(\mathbb{R}^n)\| + \left(\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M f_j \mid L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n} \right)^{1/q} \\
&< 1 + \frac{1}{j} \le 2. \qquad (18)
\end{aligned}
$$

As $\{f_j|_{B_R}\}_{j \in \mathbb{N}}$ is bounded in $B_{p,q}^s(B_R)$ and the embedding of $B_{p,q}^s(B_R)$ into $L_{\overline{p}}(B_R)$ is compact (cf. e.g. [10, Theorem 1.97, Proposition 4.6] and the references given there), $\{f_j|_{B_R}\}_{j \in \mathbb{N}}$ is precompact in $L_{\overline{p}}(B_R)$. We may assume that, for some $f \in L_{\overline{p}}(B_R)$,

$$f_j|_{B_R} \to f \quad \text{in} \quad L_{\overline{p}}(B_R), \qquad (19)$$

and, consequently,

$$\|f|L_{\overline{p}}(B_R)\| = 1. \qquad (20)$$

For $j, j' \in \mathbb{N}$, using (6), we get

$$\|f_j - f_{j'} \mid B_{p,q}^s(\mathbb{R}^n)\|$$

$$\sim \|f_j - f_{j'}|L_{\overline{p}}(\mathbb{R}^n)\| + \left(\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M (f_j - f_{j'}) \mid L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n} \right)^{1/q}.$$

Using (19) and the fact that $f_j \in \widetilde{B}_{p,q}^s(\overline{\Omega})$, we obtain

$$\|f_j - f_{j'}|L_{\overline{p}}(\mathbb{R}^n)\| = \|f_j|_{B_R} - f_{j'}|_{B_R} \,|\, L_{\overline{p}}(B_R)\| \to 0 \quad \text{as} \quad j, j' \to \infty. \qquad (21)$$

By (17) we get

$$\left(\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M (f_j - f_{j'}) \,|\, L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n} \right)^{1/q}$$

$$\lesssim \left(\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M f_j \,|\, L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n} \right)^{1/q}$$

$$+ \left(\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M f_{j'} \,|\, L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n} \right)^{1/q}$$

$$< \frac{1}{j} \, \|f_j|L_{\overline{p}}(\mathbb{R}^n)\| + \frac{1}{j'} \, \|f_{j'}|L_{\overline{p}}(\mathbb{R}^n)\|$$

$$= \frac{1}{j} + \frac{1}{j'} \to 0 \quad \text{as} \quad j, j' \to \infty.$$

Hence $\{f_j\}_{j \in \mathbb{N}}$ converges in $B_{p,q}^s(\mathbb{R}^n)$ to, say, g. As all f_j are elements of $\widetilde{B}_{p,q}^s(\overline{\Omega})$, which is a closed subspace of $B_{p,q}^s(\mathbb{R}^n)$, we get $g \in \widetilde{B}_{p,q}^s(\overline{\Omega})$. As

$$\|f_j - g|B_{p,q}^s(\mathbb{R}^n)\| \to 0 \quad \text{as} \quad j \to \infty, \qquad (22)$$

we get

$$\|f_j|_{B_R} - g|_{B_R}|L_{\overline{p}}(B_R)\| = \|f_j - g|L_{\overline{p}}(\mathbb{R}^n)\| \to 0 \quad \text{as} \quad j \to \infty \qquad (23)$$

and $g = f$ according to (19). Thus,

$$\|g|_{B_R} \,|\, L_{\overline{p}}(B_R)\| = \|f_j|_{B_R}|L_{\overline{p}}(B_R)\| = 1. \qquad (24)$$

Now (assuming $q < \infty$)

$$\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M g \,|\, L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n}$$

$$\lesssim \int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M (g - f_j) \,|\, L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n} + \int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M f_j \,|\, L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n}$$

$$\lesssim \|g - f_j|B_{p,q}^s(\mathbb{R}^n)\|^q + \frac{1}{j^q} \to 0 \quad \text{as} \quad j \to \infty.$$

Therefore

$$\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M g \,|\, L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n} = 0. \qquad (25)$$

If $q = \infty$ one has to modify the argument in the usual way. Let

$$C = \{y \in \mathbb{R}^n : g(y) \neq 0\} \cap B_R. \qquad (26)$$

6

We wish to show that $|C| = 0$. We prove it by contradiction. We suppose that $|C| > 0$. By (25), $\|\Delta_h^M g \,|\, L_p(\mathbb{R}^n)\| = 0$ for almost every $h \in \mathbb{R}^n$. Let A denote a subset of $\mathbb{R}^n$ with $|A| = 0$ such that $\|\Delta_h^M g \,|\, L_p(\mathbb{R}^n)\| = 0$, for all $h \in A^c$. We fix $h \in A^c$ such that $|h| > 2R$. As $\|\Delta_h^M g \,|\, L_p(\mathbb{R}^n)\| = 0$, there is $D_h \subset \mathbb{R}^n$ with $|D_h| = 0$ such that $(\Delta_h^M g)(x) = 0$, for all $x \in D_h^c$. As $|C| > 0$ and $|D_h| = 0$, there are points $x \in D_h^c \cap C$. Considering such an x, we get on the one hand that

$$(\Delta_h^M g)(x) = 0, \tag{27}$$

because $h \in A^c$ and $x \in D_h^c$. On the other hand, by (26) and using the hypothesis that $x \in C \subset B_R$ and $|h| > 2R$, we obtain

$$(\Delta_h^M g)(x) = \sum_{j=0}^{M} \binom{M}{j}(-1)^{M-j} g(x+jh) = (-1)^M g(x) \neq 0, \tag{28}$$

which contradicts (27). Hence $|C| = 0$, i.e., $g|_{B_R} = 0$ almost everywhere. This contradicts (24). $\qquad\square$

4 Homogeneity

Finally we present and prove the main result. Recall that $B_\lambda = \{x \in \mathbb{R}^n : |x| < \lambda\}$.

Theorem 6. *Let*

$$0 < p \leq \infty, \qquad 0 < q \leq \infty, \qquad s > \sigma_p. \tag{29}$$

Then, for all $0 < \lambda \leq 1$ and $f \in \widetilde{B}_{p,q}^s(\overline{B_\lambda})$,

$$\|f(\lambda\cdot) \,|\, B_{p,q}^s(\mathbb{R}^n)\| \sim \lambda^{s-\frac{n}{p}} \|f \,|\, B_{p,q}^s(\mathbb{R}^n)\| \tag{30}$$

where the equivalence constants are independent of λ and f.

Proof. Let us fix $0 < \lambda \leq 1$ and $f \in \widetilde{B}_{p,q}^s(\overline{B_\lambda})$. We remark that $f, f(\lambda\cdot) \in \widetilde{B}_{p,q}^s(\overline{B_1})$. By Proposition 5

$$\|f(\lambda\cdot)|B_{p,q}^s(\mathbb{R}^n)\| \sim \left(\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M(f(\lambda\cdot)) \,|\, L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n} \right)^{1/q} \tag{31}$$

uniformly in λ. One inserts

$$\Delta_h^M(f(\lambda\cdot))(x) = (\Delta_{\lambda h}^M f)(\lambda x) \tag{32}$$

in (31) and gets ($q < \infty$)

$$
\begin{aligned}
\|f(\lambda\cdot) \,|\, B_{p,q}^s(\mathbb{R}^n)\|^q &\sim \lambda^{-\frac{nq}{p}} \int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_{\lambda h}^M f|L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n} \\
&\sim \lambda^{q(s-\frac{n}{p})} \int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h^M f \,|\, L_p(\mathbb{R}^n)\|^q \frac{dh}{|h|^n} \\
&\sim \lambda^{q(s-\frac{n}{p})} \|f \,|\, B_{p,q}^s(\mathbb{R}^n)\|^q. \tag{33}
\end{aligned}
$$

If $q = \infty$ one has to modify in the usual way. $\qquad\qquad\qquad\square$

5 Relations to Hardy inequalities

Denoting the right-hand side of (10) by $\|f\,|b_{p,q}^s(\mathbb{R}^n)\|$ (which makes sense for all $0 < p \le \infty$, $0 < q \le \infty$, $0 < s < M \in \mathbb{N}$) it follows from (6) (omitting M on the left-hand side) for $\lambda > 0$ that

$$\|f(\lambda\cdot)\,|B_{p,q}^s(\mathbb{R}^n)\| = \lambda^{-\frac{n}{p}}\|f\,|L_p(\mathbb{R}^n)\| + \lambda^{s-\frac{n}{p}}\|f\,|b_{p,q}^s(\mathbb{R}^n)\|.$$

Then the equivalence (30) can be reduced to the Hardy inequality

$$\|f\,|L_p(\mathbb{R}^n)\| \le c\lambda^s\,\|f\,|B_{p,q}^s(\mathbb{R}^n)\|, \quad f \in \widetilde{B}_{p,q}^s(\overline{B_\lambda}),$$

where $c > 0$ is independent of f and of $0 < \lambda < 1$. Inequalities of this type attracted a lot of attention over the years. An early paper related to the above problems is [4]. More recent results (also for weighted counterparts) and references may be found in [2, 3].

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