Compactness in quasi-Banach function spaces and applications to compact embeddings of Besov-type spaces*

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Abstract

There are two main aims of the paper. The first one is to extend the criterion for the precompactness of sets in Banach function spaces to the setting of quasi-Banach function spaces. The second one is to extend the criterion for the precompactness of sets in the Lebesgue spaces $L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, to the so-called power quasi-Banach function spaces. These criteria are applied to establish compact embeddings of abstract Besov spaces into quasi-Banach function spaces. The results are illustrated on embeddings of Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, $0 < s < 1$, $0 < p, q \leq \infty$, into Lorentz-type spaces.

Keywords: quasi-Banach function space, compactness, compact embedding, absolute continuity, Besov space, Lorentz space

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1 Introduction

The well-known criterion for the precompactness of sets in a Banach function space states that a subset K of the absolutely continuous part X_a of a Banach function space X is precompact in X if and only if K is locally precompact in measure and K has uniformly absolutely continuous norm (cf. [3, Chap. 1, Exercise 8]).

Such a criterion was, for example, used in [21] to establish compact embeddings $W^k X(\Omega) \hookrightarrow\hookrightarrow Y(\Omega)$. Here $k \in \mathbb{N}$, Ω is a bounded domain in $\mathbb{R}^n$, $X(\Omega)$ and $Y(\Omega)$ are rearrangement-invariant Banach function spaces and $W^k X(\Omega)$ is the Sobolev space modelled upon the space $X(\Omega)$. Another paper using such a criterion is, e.g., [22], where the authors applied it to get the so-called dominated compactness theorem for regular linear integral operators.

There is a natural question whether the above mentioned criterion characterizing precompact subsets in Banach function spaces can be extended to the setting of quasi-Banach function spaces even when elements of these spaces are not locally integrable (we refer to Section 3 for the definition of quasi-Banach function spaces). The positive answer is given in Theorem 3.17 below. In particular, when the given quasi-Banach function space is the Lebesgue space $L_p(\mathbb{R}^n)$ with $0 < p < 1$, we recover [15, Lem. 1.1].

We also establish an extension of a criterion characterizing precompact sets in the Lebesgue space $L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, (cf., e.g., [1, Thm. 2.32] or [7, Thm. IV.8.21]) to the case when the space $L_p(\mathbb{R}^n)$ is replaced by a power quasi-Banach function space over $\mathbb{R}^n$ (see Theorem 4.9 and Remark 4.10; we refer to Definition 4.5 for the notion of a power quasi-Banach function space).

We apply our criteria to establish compact embeddings of abstract Besov spaces into quasi-Banach function spaces over bounded measurable subsets Ω of $\mathbb{R}^n$ (see Theorem 5.3 and Corollary 5.4; abstract Besov spaces are introduced in Definition 5.1).

Finally, we illustrate our results on embeddings of Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, $0 < s < 1$, $0 < p, q \leq \infty$, into Lorentz-type spaces over bounded measurable subsets of $\mathbb{R}^n$ (see Theorem 6.3; we refer to Section 2 for the definition of Lorentz-type spaces).

2 Notation and Preliminaries

For two non-negative expressions $\mathcal{A}$ and $\mathcal{B}$, the symbol $\mathcal{A} \lesssim \mathcal{B}$ (or $\mathcal{A} \gtrsim \mathcal{B}$) means that $\mathcal{A} \leq c\mathcal{B}$ (or $c\mathcal{A} \geq \mathcal{B}$), where c is a positive constant independent of appropriate quantities involved in $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{A} \gtrsim \mathcal{B}$, we write $\mathcal{A} \approx \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are *equivalent*.

Given a set A, its characteristic function is denoted by χ_A. For $a \in \mathbb{R}^n$ and $r \geq 0$, the symbol $B(a, r)$ stands for the closed ball in $\mathbb{R}^n$ centred at a with the radius r. The notation $|\cdot|_n$ is used for Lebesgue measure in $\mathbb{R}^n$.

Let (R,μ) be a measure space (with a non-negative measure μ).[1] The symbol $\mathcal{M}(R,\mu)$ is used to denote the family of all complex-valued or extended real-valued μ-measurable functions defined μ-a.e. on R. By $\mathcal{M}^+(R,\mu)$ we mean the subset of $\mathcal{M}(R,\mu)$ consisting of those functions which are non-negative μ-a.e. on R. If R is a measurable subset Ω of $\mathbb{R}^n$ and μ is the corresponding Lebesgue measure, we omit the μ from the notation. Moreover, if $\Omega=(a,b)$, we write simply $\mathcal{M}(a,b)$ and $\mathcal{M}^+(a,b)$ instead of $\mathcal{M}(\Omega)$ and $\mathcal{M}^+(\Omega)$, respectively. Finally, by $\mathcal{W}(\Omega)$ (or by $\mathcal{W}(a,b)$) we mean the class of *weight functions* on Ω (resp. on (a,b)), consisting of all measurable functions which are positive a.e. on Ω (resp. on (a,b)). A subscript 0 is added to the previous notation (as in $\mathcal{M}_0(\Omega)$, for example) if in the considered class one restricts to functions which are finite a.e..

Given two quasi-normed spaces X and Y, we write $X=Y$ (and say that X and Y *coincide*) if X and Y are equal in the algebraic and the topological sense (their quasi-norms are equivalent).

Let $p,q\in(0,\infty]$, let Ω be a measurable subset of $\mathbb{R}^n$ with $|\Omega|_n>0$ and let $w\in\mathcal{W}(0,|\Omega|_n)$ be such that

$$\begin{aligned} B_{p,q;w}(t): \;&=\; \|\tau^{1/p-1/q}\, w(\tau)\|_{q;(0,t)} \;<\; \infty \quad \text{for all } t\in(0,|\Omega|_n) \\ &\quad\;\text{(and also for } t=|\Omega|_n \text{when } |\Omega|_n<\infty), \end{aligned} \tag{1}$$

where $\|\cdot\|_{q;E}$ is the usual L_q-(quasi-)norm on the measurable set E. The *Lorentz-type space* $L_{p,q;w}(\Omega)$ consists of all (classes of) functions $f\in\mathcal{M}(\Omega)$ for which the quantity

$$\|f\|_{p,q;w;\Omega} := \|t^{1/p-1/q}\, w(t)\, f^*(t)\|_{q;(0,|\Omega|_n)} \tag{2}$$

is finite; here f^* denotes the *non-increasing rearrangement* of f given by

$$f^*(t)=\inf\{\lambda>0: |\{x\in\Omega:\ |f(x)|>\lambda\}|_n\le t\},\quad t\ge 0. \tag{3}$$

We shall also need the inequality (cf. [3, p. 41])

$$(f+g)^*(t)\le f^*(t/2)+g^*(t/2),\quad t\ge 0, \tag{4}$$

and the maximal function f^{**} of f^* defined by

$$f^{**}(t)=\frac{1}{t}\int_0^t f^*(s)\,ds,\quad t>0.$$

One can show that the functional (2) is a *quasi-norm* on $L_{p,q;w}(\Omega)$ *if and only if* the function $B_{p,q;w}$ given by (1) satisfies

$$B_{p,q;w}\in\Delta_2, \tag{5}$$

that is, $B_{p,q;w}(2t)\lesssim B_{p,q;w}(t)$ for all $t\in(0,|\Omega|_n/2)$. This follows, e.g., from [5, Cor. 2.2] if $q\in(0,\infty)$. When $q=\infty$, then arguments similar to those used in the proof of [5, Cor. 2.2] together with inequality (4) and the fact that

$$\|f\|_{p,\infty;w;\Omega}=\|B_{p,\infty;w}(t)f^*(t)\|_{\infty,(0,|\Omega|_n)}\quad \text{for all } f\in L_{p,\infty;w}(\Omega) \tag{6}$$

[1] Here we use notation from [3]. To be more precise, instead of (R,μ), one should write (R,Σ,μ), where Σ is a σ-algebra of μ-measurable subsets of R.

yield the result. Note also that equality (6) follows on interchanging the essential suprema on its right-hand side. In particular, one can easily verify that (5) is satisfied provided that

$$w(2t) \lesssim w(t) \quad \text{for a.e. } t \in (0, |\Omega|_n/2).$$

Moreover, since the relation $w \in \mathcal{W}(0, |\Omega|_n)$ yields $B_{p,q;w}(t) > 0$ for all $t \in (0, |\Omega|_n)$, one can prove that the space $L_{p,q;w}(\Omega)$ is *complete* when (5) is satisfied (cf. [4, Prop. 2.2.9]; if $q = \infty$ one makes use of (6) again).

If $q \in [1, \infty)$, the spaces $L_{p,q;w}(\Omega)$ are particular cases of the *classical Lorentz spaces* $\Lambda^q(\omega)$. On the other hand, these Lorentz-type spaces contain as particular cases a lot of well-known spaces such as Lebesgue spaces $L_p(\Omega)$ and Lorentz spaces $L_{p,q}(\Omega)$, among others.

If $\Omega = \mathbb{R}^n$, we sometimes omit this symbol in the notation and, for example, simply write $\|\cdot\|_{p,q;w}$ or $L_{p,q;w}$ instead of $\|\cdot\|_{p,q;w;\mathbb{R}^n}$ or $L_{p,q;w}(\mathbb{R}^n)$, respectively.

3 A compactness criterion in quasi-Banach function spaces

In what follows, the symbol (R, μ) stands for a totally σ-finite measure space and in $\mathcal{M}_0 = \mathcal{M}_0(R, \mu)$ we consider the topology of convergence in measure on sets of finite measure (see [3, p. 3]), which we briefly refer to as the topology of local convergence in measure.

Definition 3.1. A mapping $\rho : \mathcal{M}^+(R, \mu) \to [0, \infty]$ is called a *function quasi-norm* if there exists a constant $C \in [1, \infty)$ such that, for all f, g, f_k ($k \in \mathbb{N}$) in $\mathcal{M}^+(R, \mu)$, for all constants $a \geq 0$ and for all μ-measurable subsets E of R, the following properties hold:

(P1) $\rho(f) = 0 \Leftrightarrow f = 0$ μ-a.e.; $\quad \rho(af) = a\rho(f); \quad \rho(f+g) \leq C\,(\rho(f)+\rho(g))$;
(P2) $g \leq f$ μ-a.e. $\Rightarrow$ $\rho(g) \leq \rho(f)$;
(P3) $f_k \uparrow f$ μ-a.e. $\Rightarrow$ $\rho(f_k) \uparrow \rho(f)$;
(P4) $\mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty$.

Definition 3.2. Let $\rho : \mathcal{M}^+(R, \mu) \to [0, \infty]$ satisfy properties **(P1)-(P3)** of Definition 3.1. The collection $X = X(\rho)$ of all functions $f \in \mathcal{M}(R, \mu)$ for which $\|f\|_X := \rho(|f|) < \infty$ is called a *quasi-Banach function lattice* (*q-BFL*, for short) over (R, μ) (or, simply, over R if μ is clearly meant). If, in addition, ρ is a function quasi-norm, then we also call $X(\rho)$ a *quasi-Banach function space* (*q-BFS*, for short) over (R, μ) (or, simply, over R).

In what follows we shall use the fact that in any quasi-normed linear space $(X, \|\cdot\|_X)$ there is a λ-norm $|||\cdot|||$ (with $\lambda \in (0, 1]$ satisfying $(2C)^\lambda = 2$, where C is from Definition 3.1) such that

$$|||f||| \approx \|f\|_X \quad \text{for all} \quad f \in X \tag{7}$$

– cf., e.g., [9, p. 2] and [6, p. 20]. This result goes back to [2] and [23] – see also [20, pp. 66-67]. Recall that the λ-norm $|||\cdot|||$, $\lambda\in(0,1]$, satisfies, for all $f,g\in X$ and all scalars α,

$$|||f|||=0 \quad \text{if and only if} \quad f=0,$$

$$|||\alpha f|||=|\alpha|\ |||f|||,$$
$$|||f+g|||^{\lambda}\le|||f|||^{\lambda}+|||g|||^{\lambda}. \tag{8}$$

Lemma 3.3. *Let $X=X(\rho)$ be a q-BFL. Then $X\subset\mathcal{M}_0$ and under the natural vector space operations $(X,\|\cdot\|_X)$ is indeed a quasi-normed linear space. Moreover, $X\hookrightarrow\mathcal{M}_0$. In particular, if $f_k\underset{k}{\longrightarrow}f$ in X, then $f_k\underset{k}{\longrightarrow}f$ locally in measure, and hence some subsequence converges pointwise μ-a.e. to f.*

Proof. Given any $f\in X$, the set A in which f is infinite has a measure 0. Indeed, as $N\chi_A\le|f|$ for any $N\in\mathbb{N}$, properties **(P1)** and **(P2)** imply that

$$N\rho(\chi_A)=\rho(N\chi_A)\le\rho(|f|)<\infty,$$

and thus $\rho(\chi_A)=0$. Together with **(P1)**, this shows that $\chi_A=0$ μ-a.e..

Since $\mu(A)=0$, X is a subspace of $\mathcal{M}_0$ and so X inherits the vector space operations from $\mathcal{M}_0$ (where, as usual, any two functions coinciding μ-a.e. are identified). Moreover, by **(P1)**, the space $(X,\|\cdot\|_X)$ is a quasi-normed linear space.

It remains to prove the continuous embedding $X\hookrightarrow\mathcal{M}_0$. This can be done using some ideas of [16, Chap. II, Thm. 1, pp. 41-42]. However, since our setting and that of [16] are different, we prove it here for the convenience of the reader.

It is sufficient to show that the condition $\|f_k\|_X\underset{k}{\longrightarrow}0$ implies the convergence of $\{f_k\}_k$ to zero in $\mathcal{M}_0$.

On the contrary, assume that there are a set E, with $0<\mu(E)<\infty$, and $\varepsilon>0$ such that $\mu\{x\in E:\ |f_k(x)|>\varepsilon\}$ fails to converge to 0 as $k\to\infty$. Then there exists $\delta>0$ and a subsequence $\{f_{\sigma(k)}\}_k$ such that the inequalities $|f_{\sigma(k)}(x)|>\varepsilon$ hold on sets $E_k\subset E$ satisfying $\mu(E_k)>\delta$ for all $k\in\mathbb{N}$. Hence, $\varepsilon\chi_{E_k}(x)\le|f_{\sigma(k)}(x)|$, and, on using **(P2)** and **(P1)**, we obtain that $\varepsilon\|\chi_{E_k}\|_X\le\|f_{\sigma(k)}\|_X$ for all $k\in\mathbb{N}$. Without loss of generality, we may assume that the sequence $\{\sigma(k)\}_k$ is chosen so that $\sum_{k=1}^{\infty}\|f_{\sigma(k)}\|_X^{\lambda}<\infty$, where $\lambda\in(0,1]$ corresponds to the λ-norm $|||\cdot|||$ considered in (7). Consequently,

$$\begin{aligned}\varepsilon^{\lambda}\Big\|\sum_{k=1}^{N}\chi_{E_k}\Big\|_X^{\lambda}&\approx\varepsilon^{\lambda}\Big|\Big|\Big|\sum_{k=1}^{N}\chi_{E_k}\Big|\Big|\Big|^{\lambda}\ \lesssim\ \varepsilon^{\lambda}\sum_{k=1}^{N}\|\chi_{E_k}\|_X^{\lambda}\\&\le\ \sum_{k=1}^{N}\|f_{\sigma(k)}\|_X^{\lambda}\ \le\ \sum_{k=1}^{\infty}\|f_{\sigma(k)}\|_X^{\lambda}\ <\ \infty\ \text{ for all } N\in\mathbb{N}\end{aligned}\tag{9}$$

(we emphasize that constants hidden in symbols $\approx$ and $\lesssim$ are independent of N). Thus, if we show that

$$\Big\| \sum_{k=1}^{N} \chi_{E_k} \Big\|_X \underset{N}{\longrightarrow} \infty, \tag{10}$$

we arrive at a contradiction and the proof will be complete.

In order to prove (10), it is enough to verify that

$$\sum_{k=1}^{N} \chi_{E_k} \uparrow f \quad \text{with} \quad f \notin \mathcal{M}_0. \tag{11}$$

Indeed, (11) and **(P3)** imply that $\| \sum_{k=1}^{N} \chi_{E_k} \|_X \uparrow \rho(f)$ and, since $f \notin \mathcal{M}_0 \supset X$, $\rho(f) = \infty$. As it is obvious that the sequence $\{\sum_{k=1}^{N} \chi_{E_k}\}_N$ is non-decreasing, all that remains to prove in order to establish (11) is that $f \notin \mathcal{M}_0$.

Again we proceed by contradiction and assume that $f \in \mathcal{M}_0$. Recalling that $E_k \subset E$, $k \in \mathbb{N}$, and $\mu(E) < \infty$, we see that we can use Egorov's Theorem to state that there exists a set $E' \subset E$, with $\mu(E \setminus E') < \frac{\delta}{2}$, on which the convergence of $\sum_{k=1}^{N} \chi_{E_k}$ to f is uniform. As a consequence, the boundedness of each $\sum_{k=1}^{N} \chi_{E_k}$ on E' implies that f is bounded on E', too. Thus,

$$\int_{E'} f \, d\mu < \infty. \tag{12}$$

However, since the inequalities $\mu(E_k \setminus E') < \frac{\delta}{2}$ and $\mu(E_k) > \delta$ imply that $\mu(E' \cap E_k) > \frac{\delta}{2}$, $k \in \mathbb{N}$, we arrive at

$$\int_{E'} f \, d\mu \geq \int_{E'} \sum_{k=1}^{N} \chi_{E_k} \, d\mu = \sum_{k=1}^{N} \mu(E' \cap E_k) > N \frac{\delta}{2}$$

for all $N \in \mathbb{N}$, which contradicts (12). □

Remark 3.4. (i) In contrast to [19, p. 9], in our definition of a q-BFS we do not require *a priori* that $X \hookrightarrow \mathcal{M}_0$ and that X is complete since these two properties are consequences of axioms **(P1)-(P4)**(cf. Lemma 3.3 and Lemma 3.6 below).

(ii) Recall that (cf. [3]) a *function norm* is a mapping $\rho : \mathcal{M}^+(R, \mu) \to [0, \infty]$ satisfying **(P1)** with $C = 1$, **(P2)-(P4)** and

(P5) $\mu(E) < \infty \Rightarrow \int_E f \, d\mu \leq c_E \, \rho(f)$ for some constant c_E, $0 < c_E < \infty$, depending on E and ρ but independent of $f \in \mathcal{M}^+(R, \mu)$.

Thus, any function norm is a function quasi-norm. Hence, taking a function norm ρ and defining the Banach function space (BFS) as the family of those $f \in \mathcal{M}(R, \mu)$ for which $\rho(|f|) < \infty$, we see that any BFS is a q-BFS.

The first example of q-BFS (and, *a fortiori*, of q-BFL) is the Lebesgue space $L_p(\Omega)$ with $0 < p \leq \infty$, which is also a BFS when $p \geq 1$. The next example of q-BFS is the Lorentz-type space $L_{p,q;w}(\Omega)$ introduced in Section 2 provided that (1) and (5) hold. (In this case $R = \Omega$, μ is the Lebesgue measure on Ω and $\rho = \|\cdot\|_{p,q;w;\Omega}$.) Indeed, property **(P1)** follows from what has been said in Section 2 and properties **(P2)-(P4)** are easy consequences of the definition of such spaces.

The next result follows by the same arguments as [3, Chap. 1, Lem. 1.5].

Lemma 3.5. *Let $X = X(\rho)$ be a q-BFL and assume that $f_k \in X$, $k \in \mathbb{N}$.*
(i) *If $0 \le f_k \uparrow f$ μ-a.e., then either $f \notin X$ and $\|f_k\|_X \uparrow \infty$, or $f \in X$ and $\|f_k\|_X \uparrow \|f\|_X$.*
(ii) (Fatou's lemma) *If $f_k \to f$ μ-a.e., and if $\liminf_{k\to\infty} \|f_k\| < \infty$, then $f \in X$ and $\|f\|_X \le \liminf_{k\to\infty} \|f_k\|_X$.*

We shall also need the following counterpart of [3, Chap. 1, Thm. 1.6], where the number $\lambda \in (0,1]$ corresponds to the λ-norm $|||\cdot|||$ appearing in (7).

Lemma 3.6. *Let $X = X(\rho)$ be a q-BFL. Assume that $f_k \in X$, $k \in \mathbb{N}$, and that*

$$\sum_{k=1}^{\infty} \|f_k\|_X^{\lambda} < \infty. \tag{13}$$

Then $\sum_{k=1}^{\infty} f_k$ converges in X to a function $f \in X$ and

$$\|f\|_X \lesssim \left(\sum_{k=1}^{\infty} \|f_k\|_X^{\lambda} \right)^{1/\lambda}. \tag{14}$$

In particular, X is complete.

Proof. If $t = \sum_{k=1}^{\infty} |f_k|$, $t_K = \sum_{k=1}^{K} |f_k|$, $K \in \mathbb{N}$, then $0 \le t_K \uparrow t$. Since, by (7) and (8),

$$\|t_K\|_X \lesssim \left(\sum_{k=1}^{K} \|f_k\|_X^{\lambda} \right)^{1/\lambda} \quad \text{for all} \quad K \in \mathbb{N},$$

it follows from (13) and Lemma 3.5(i) that $t \in X$. In particular, since $X \subset \mathcal{M}_0$, $\sum_{k=1}^{\infty} |f_k(x)|$ converges pointwise μ-a.e., and hence so does $\sum_{k=1}^{\infty} f_k(x)$. Thus, if

$$f := \sum_{k=1}^{\infty} f_k, \quad s_K := \sum_{k=1}^{K} f_k, \quad K \in \mathbb{N},$$

then $s_K \to f$ μ-a.e.. Hence, given any $M \in \mathbb{N}$, $s_K - s_M \to f - s_M$ μ-a.e. as $K \to \infty$. Moreover,

$$\liminf_{K\to\infty} \|s_K - s_M\|_X \lesssim \liminf_{K\to\infty} \left(\sum_{k=M+1}^{K} \|f_k\|_X^{\lambda} \right)^{1/\lambda} = \left(\sum_{k=M+1}^{\infty} \|f_k\|_X^{\lambda} \right)^{1/\lambda},$$

which tends to 0 as $M \to \infty$, because of hypothesis (13). Then, by Fatou's lemma (Lemma 3.5(ii)), $f - s_M \in X$ (therefore also $f \in X$) and $\|f - s_M\|_X \to 0$ as $M \to \infty$. Finally,

$$\|f\|_X = \|f - s_M + \sum_{k=1}^{M} f_k\|_X \lesssim \left(\|f - s_M\|_X^\lambda + \sum_{k=1}^{M} \|f_k\|_X^\lambda \right)^{1/\lambda}$$

and (14) follows by letting M tend to infinity.

Completeness follows by standard arguments as in the normed case. □

As a consequence of this lemma, a q-BFL is a quasi-Banach space (i.e., a complete quasi-normed space).

Definition 3.7. A sequence $\{E_n\}_n$ of μ-measurable subsets of a measure space (R, μ) is said to converge μ-a.e. to the empty set (notation $E_n \underset{n}{\longrightarrow} \emptyset$ μ-a.e.) if $\cap_{n=1}^{\infty} \cup_{m=n}^{\infty} E_m$ is a set of μ-measure zero or, equivalently, if $\chi_{E_n} \underset{n}{\longrightarrow} 0$ μ-a.e.

Definition 3.8. A function f in a q-BFL X is said to have *absolutely continuous quasi-norm* if $\|f\chi_{E_n}\|_X \underset{n}{\longrightarrow} 0$ for every sequence $\{E_n\}_{n=1}^{\infty}$ such that $E_n \underset{n}{\longrightarrow} \emptyset$ μ-a.e.. The set of all functions in X which have absolutely continuous quasi-norm is denoted by X_a. If $X_a = X$, then the space X is said to have *absolutely continuous quasi-norm.*

Analogously to the BFS case, one can prove the next assertion (cf. [3, Chap. 1, Prop. 3.2].

Proposition 3.9. *A function f in a q-BFL X has absolutely continuous quasi-norm if and only if $\|f\chi_{E_n}\|_X \downarrow 0$ for every sequence $\{E_n\}_{n=1}^{\infty}$ such that $E_n \downarrow \emptyset$ μ-a.e. (which means, besides $E_n \underset{n}{\longrightarrow} \emptyset$ μ-a.e., that the sequence $\{E_n\}_{n=1}^{\infty}$ is non-increasing).*

Example 3.10. If $0 < p \leq \infty$, $0 < q < \infty$ and the weight w satisfies (1) and (5), then the space $L_{p,q;w}(\Omega)$ has absolutely continuous quasi-norm. This follows immediately from Proposition 3.9 and the Lebesgue dominated convergence theorem.

Proposition 3.11. *The set X_a of functions from the q-BFL X with absolutely continuous quasi-norm is a closed linear subspace of X.*

The proof follows by the same arguments as those of [3, Chap. 1, Thm. 3.8].

Definition 3.12. Let X be a q-BFL and let $K \subset X_a$. Then K is said to have *uniformly absolutely continuous quasi-norm* in X (notation $K \subset \mathrm{UAC}(X)$) if, for every sequence $\{E_n\}_{n=1}^{\infty}$ with $E_n \underset{n}{\longrightarrow} \emptyset$ μ-a.e.,

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N}:\ f \in K,\ n \geq N \ \Rightarrow \|f\chi_{E_n}\|_X < \varepsilon.$$

As in Definition 3.8, it is irrelevant in Definition 3.12 whether we consider all sequences of sets satisfying $E_n \underset{n}{\longrightarrow} \emptyset$ or $E_n \downarrow \emptyset$ μ-a.e..

The following assertion is a consequence of Definition 3.12.

Proposition 3.13. *If $K \subset UAC(X)$, then*

$$\forall \varepsilon > 0, \exists \delta > 0 : \ f \in K, \ \mu(E) < \delta \ \Rightarrow \ \|f\chi_E\|_X < \varepsilon. \tag{15}$$

Proof. Assume that $K \subset \mathrm{UAC}(X)$ and that (15) is false. Then there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ there exists $f_n \in K$ and $E_n \subset R$ satisfying

$$\mu(E_n) < 2^{-n} \quad \text{and} \quad \|f_n \chi_{E_n}\|_X \geq \varepsilon. \tag{16}$$

Hence, $\mu(\cup_{n=m}^{\infty} E_n) \leq \sum_{n=m}^{\infty} \mu(E_n) < 2^{-m+1}$, and thus $E_n \to \emptyset$ μ-a.e.. Together with the fact that $K \subset \mathrm{UAC}(X)$, this shows that there is $N \in \mathbb{N}$ such that $\|f_n \chi_{E_n}\|_X < \varepsilon$ for all $n \geq N$. However, this contradicts (16) and the result follows. □

Remark 3.14. (i) When R has finite measure, the converse of the preceding result is also true.

(ii) For the sake of completeness, let us mention that, similarly to Proposition 3.13, any function f with *absolutely continuous quasi-norm* in a q-BFL X satisfies

$$\forall \varepsilon > 0, \exists \delta > 0 : \ \mu(E) < \delta \ \Rightarrow \ \|f\chi_E\|_X < \varepsilon.$$

The converse is also true when R has finite measure.

Lemma 3.15. *Let X be a q-BFS. The sequence $\{f_k\}_k \subset X_a$ is convergent in X if and only if $\{f_k\}_k$ converges locally in measure and $\{f_k\}_k \subset UAC(X)$. Moreover, in the case of convergence the two limits coincide.*

Proof. Assume that $\{f_k\}_k \subset X_a$ and that it converges in quasi-norm to $f \in X$. Then it follows from Lemma 3.3 that $\{f_k\}_k$ converges locally in measure to f.

We now show that $\{f_k\}_k \subset \mathrm{UAC}(X)$. Let $\varepsilon > 0$ and consider any $\{E_m\}_{n=1}^{\infty}$ with $E_m \underset{m}{\longrightarrow} \emptyset$ μ-a.e.. As $f_k \underset{k}{\longrightarrow} f$ in X, there exists $N \in \mathbb{N}$ such that

$$\|f_k - f\|_X < \frac{\varepsilon}{3C^2} \quad \text{ for all } \ k \geq N, \tag{17}$$

where C is the constant from Definition 3.1. Since $\{f_k\}_k \subset X_a$, there exists $M \in \mathbb{N}$ such that

$$\|f_k \chi_{E_m}\|_X < \frac{\varepsilon}{3C^2} < \varepsilon \quad \text{for all } \ m \geq M \ \text{ and } \ k = 1, \ldots, N.$$

Together with (17) and properties **(P1)** and **(P2)** this implies that

$$\begin{aligned} \|f_k \chi_{E_m}\|_X &\leq C\|(f_k - f)\chi_{E_m}\|_X + C\|f\chi_{E_m}\|_X \\ &\leq C\|f_k - f\|_X + C^2\|(f - f_N)\chi_{E_m}\|_X + C^2\|f_N \chi_{E_m}\|_X \\ &< \frac{\varepsilon}{3C} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \ \leq \varepsilon \quad \text{for all } m \geq M \ \text{ and } \ k > N. \end{aligned}$$

Thus, $\{f_k\}_k \subset \mathrm{UAC}(X)$.

Conversely, assume now that $\{f_k\}_k$ converges locally in measure to some function $f \in \mathcal{M}_0(R,\mu)$ and that $\{f_k\}_k \subset \mathrm{UAC}(X)$. We start by observing that the first of these two hypotheses guarantees that $f_{\sigma(k)} \underset{k}{\longrightarrow} f$ μ-a.e. for some subsequence $\{f_{\sigma(k)}\}_k$ of $\{f_k\}_k$. Let $\varepsilon > 0$. Since our measure space (R,μ) is totally σ-finite, there is a non-decreasing sequence of sets F_n such that $\cup_{n=1}^{\infty} F_n = R$ and $\mu(F_n) < \infty$ for all $n \in N$. Without loss of generality, we can assume that $0 < \mu(F_n)$, $n \in N$. Then $(R \setminus F_n) \underset{n}{\longrightarrow} \emptyset$ μ-a.e. and the hypothesis $\{f_k\}_k \subset \mathrm{UAC}(X)$ implies that there exists $N \in \mathbb{N}$ such that $\|f_k \chi_{R\setminus F_n}\|_X < \frac{\varepsilon}{6C^2}$ for all k and all $n \geq N$. Hence, on putting $P := F_N$, we have $\mu(P) < \infty$ and

$$\|f_k \chi_{R\setminus P}\|_X < \frac{\varepsilon}{6C^2} \quad \text{for all } k \in \mathbb{N}. \tag{18}$$

Then Fatou's lemma applied to $f_{\sigma(k)}\chi_{R\setminus P}$ implies that also

$$f\chi_{R\setminus P} \in X \quad \text{and} \quad \|f\chi_{R\setminus P}\|_X \leq \frac{\varepsilon}{6C^2}. \tag{19}$$

On the other hand, together with property **(P4)** applied to P, the convergence of $\{f_k\}_k$ locally in measure to f guarantees that

$$\mu(E_j) \underset{j}{\longrightarrow} 0, \quad \text{where } E_j := \{x \in P : |f_j(x) - f(x)| > \frac{\varepsilon}{3C^2\rho(\chi_P)}\}. \tag{20}$$

Now the hypothesis that $\{f_k\}_k \subset \mathrm{UAC}(X)$, Proposition 3.13 and (20) imply that there exists $J \in \mathbb{N}$ such that

$$\|f_k \chi_{E_j}\|_X < \frac{\varepsilon}{6C^3} \quad \text{for all } k \in \mathbb{N} \text{ and all } j \geq J. \tag{21}$$

Thus, another application of Fatou's lemma, to $f_{\sigma(k)}\chi_{E_j}$, allows us to conclude that also

$$f\chi_{E_j} \in X \quad \text{and} \quad \|f\chi_{E_j}\|_X \leq \frac{\varepsilon}{6C^3} \quad \text{for all } j \geq J. \tag{22}$$

Estimates (18)-(22) and properties **(P1)** and **(P2)** imply that

$$\begin{aligned}
\rho(|f_k - f|) &\leq C\,\|(f_k - f)\chi_{R\setminus P}\|_X + C^2\|(f_k - f)\chi_{E_k}\|_X + C^2\rho(|f_k - f|\chi_{P\setminus E_k}) \\
&< C^2\,(\frac{\varepsilon}{6C^2} + \frac{\varepsilon}{6C^2}) + C^3\,(\frac{\varepsilon}{6C^3} + \frac{\varepsilon}{6C^3}) + C^2\frac{\varepsilon}{3C^2\rho(\chi_P)}\rho(\chi_P) \\
&= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \text{for all } k \geq J.
\end{aligned}$$

Therefore, $f \in X$ and $\{f_k\}_k$ converges to f in X. □

Remark 3.16. (i) The hypothesis $\{f_k\}_k \subset X_a$ was not necessary in the proof of the assertion that convergence in quasi-norm implies convergence in measure.

(ii) This lemma with $\mu(R) < \infty$ for the Orlicz norm was proved in [17, Lem. 11.2].

As already mentioned in the Introduction, in the context of Banach function spaces the following theorem is contained in [3, Chap. 1, Exercise 8].

Theorem 3.17. *Let X be a q-BFS and $K \subset X_a$. Then K is precompact in X if and only if it is locally precompact in measure and $K \subset UAC(X)$.*

Proof. Since the topologies involved are metrizable, we can prove precompactness using the notion of sequential precompactness.

The sufficiency follows immediately from Lemma 3.15.

As for the necessity, observe that if K is precompact in the space X, then, by Lemma 3.15, it is locally precompact in measure. Thus, it only remains to show that $K \subset \mathrm{UAC}(X)$. Assume that it is not the case. Then there exists a sequence of sets $E_n \downarrow \emptyset$ μ-a.e. and $\varepsilon > 0$ such that for each $k \in \mathbb{N}$ there exists a function $f_k \in K$ satisfying

$$\|f_k \chi_{E_k}\|_X \geq \varepsilon. \tag{23}$$

On the other hand, by the precompactness of K in X, there is a subsequence $\{f_{\sigma(k)}\}_k$ which converges in X, say, to f. Since $K \subset X_a$ and X_a is a closed subspace of X (cf. Proposition 3.11), the function f also has absolutely continuous quasi-norm and therefore $\|f \chi_{E_k}\|_X \underset{k}{\longrightarrow} 0$. But then

$$\|f_{\sigma(k)} \chi_{E_{\sigma(k)}}\|_X \leq C \left(\|f_{\sigma(k)} \chi_{E_{\sigma(k)}} - f \chi_{E_{\sigma(k)}}\|_X + \|f \chi_{E_{\sigma(k)}}\|_X \right) \underset{k}{\longrightarrow} 0,$$

which contradicts (23). □

Remark 3.18. Taking into account Remark 3.16, for future reference we would like to note here that the hypothesis $K \subset X_a$ was not needed in the proof of the assertion that precompactness in quasi-norm implies local precompactness in measure.

The following is an immediate consequence of Theorem 3.17. Note that the terminology "operator locally compact in measure" means an operator for which the image of the closed unit ball is precompact in the topology of local convergence in measure.

Corollary 3.19. *Let X be a quasi-normed space, Y be q-BFS and T be a linear operator acting from X into Y_a. The operator T is compact if and only if it is locally compact in measure and $\{Tf : \|f\|_X \leq 1\} \subset UAC(Y)$.*

4 A compactness criterion in quasi-Banach function spaces over $\mathbb{R}^n$

Definition 4.1. A function quasi-norm ρ over a totally σ-finite measure space (R, μ) is said to be rearrangement-invariant if $\rho(f) = \rho(g)$ for every pair of equimeasurable functions f and g in $\mathcal{M}_0^+(R, \mu)$. The corresponding q-BFS space $X = X(\rho)$ is then said to be *rearrangement-invariant* (*r.i.*, for short).

In what follows we shall often consider a q-BFS over $\mathbb{R}^n$ with the Lebesgue measure and we shall denote it simply by $L(\mathbb{R}^n)$.

Lemma 4.2. (Minkowski-type inequality) *Let $L = L(\mathbb{R}^n)$ be a BFS. If $f \in \mathcal{M}^+(\mathbb{R}^n \times \mathbb{R}^n)$ is such that $\int_{\mathbb{R}^n} \|f(\cdot, y)\|_L \, dy < \infty$, then*

$$\left\| \int_{\mathbb{R}^n} f(\cdot, y)\, dy \right\|_L \le \int_{\mathbb{R}^n} \|f(\cdot, y)\|_L \, dy.$$

Proof. Using the properties of the norm in L and Fubini Theorem, we obtain

$$\begin{aligned} \left\| \int_{\mathbb{R}^n} f(\cdot, y)\, dy \right\|_L &= \sup_{\|g\|_{L'} \le 1} \Big| \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} f(x, y)\, dy \Big) g(x)\, dx \Big| \\ &\le \sup_{\|g\|_{L'} \le 1} \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f(x, y) g(x)|\, dx \Big)\, dy \\ &\le \sup_{\|g\|_{L'} \le 1} \int_{\mathbb{R}^n} \|f(\cdot, y)\|_L \|g\|_{L'}\, dy \\ &\le \int_{\mathbb{R}^n} \|f(\cdot, y)\|_L \, dy \end{aligned}$$

(where L' stands for the associate space of L). □

Theorem 4.3. *Let $L = L(\mathbb{R}^n)$ be a BFS.*
(a) *If L is r.i. and $K \subset L_a$ is precompact in L, then:*
 (i) *K is bounded in L;*
 (ii) *$\forall \varepsilon > 0, \exists$compact $G \subset \mathbb{R}^n : \forall u \in K$, $\|u\chi_{\mathbb{R}^n \setminus G}\|_L < \varepsilon$;*
 (iii) *$\forall \varepsilon > 0, \exists \delta > 0 : \forall u \in K$, $|h| < \delta \Rightarrow \|\Delta_h u\|_L < \varepsilon$.* [2]
(b) *Conversely, if $K \subset L$ satisfies conditions* (i)-(iii), *then K is precompact in L.*

Proof. With appropriate modifications, we follow the arguments which prove the well-known characterization of the compactness in Lebesgue spaces (cf., e.g., [1, Thm. 2.32]).

Suppose first that $K \subset L_a$ is precompact. Then, given $\varepsilon > 0$, there exists a finite set $N_\varepsilon \subset K$ such that

$$K \subset \bigcup_{f \in N_\varepsilon} B_{\varepsilon/6}(f),$$

where $B_r(f)$ stands for the open ball in L of radius $r > 0$ and centred at f. Since $\overline{C_0^\infty(\mathbb{R}^n)} \supset L_a$ (cf. [8, Rem. 3.13]) and, by hypothesis, $L_a \supset K$, there exists a finite set S of continuous functions with compact support in $\mathbb{R}^n$ such that for each $u \in K$ there exists $\phi \in S$ satisfying $\|u - \phi\|_L < \varepsilon/3$. Let G be the union of the supports of the finitely many functions from S. Then $G \subset\subset \mathbb{R}^n$ and

$$\|u\chi_{\mathbb{R}^n \setminus G}\|_L = \|u\chi_{\mathbb{R}^n \setminus G} - \phi\chi_{\mathbb{R}^n \setminus G}\|_L \le \|u - \phi\|_L < \varepsilon/3 \quad \text{for all } u \in K.$$

[2]Recall that $\Delta_h u(x) := u(x+h) - u(x)$, $x \in \mathbb{R}^n$.

Since continuous functions with compact supports in $\mathbb{R}^n$ are L-mean continuous [3] and the set S is finite, there exists $\delta > 0$ such that

$$\|\phi(\cdot + h) - \phi(\cdot)\|_L < \varepsilon/3 \quad \text{when } |h| < \delta \text{ and } \phi \in S.$$

Then, for any $u \in K$ and $\phi \in S$ such that $\|u - \phi\|_L < \varepsilon/3$,

$$\begin{aligned}
&\|u(\cdot + h) - u(\cdot)\|_L \\
&\quad \leq \|u(\cdot + h) - \phi(\cdot + h)\|_L + \|\phi(\cdot + h) - \phi(\cdot)\|_L + \|\phi(\cdot) - u(\cdot)\|_L \\
&\quad < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \quad \text{if } |h| < \delta
\end{aligned}$$

(we have also used the fact that L is r.i.).

We have shown that conditions (ii) and (iii) hold. Of course, being precompact, K is also bounded (in L). Therefore (i) is also verified.

We now prove the converse result, i.e., that conditions (i)-(iii) are sufficient for the precompactness of K in L.

Let $\varepsilon > 0$ be given. By condition (ii), there is a compact set G in $\mathbb{R}^n$ such that

$$\|u\chi_{\mathbb{R}^n \setminus G}\|_L < \varepsilon/3 \quad \text{for all } u \in K. \tag{24}$$

Let J be a non-negative function in $C_0^\infty(\mathbb{R}^n)$ satisfying $J(x) = 0$ if $|x| \geq 1$ and $\int_{\mathbb{R}^n} J(x)\,dx = 1$. Put $J_\eta(x) = \eta^{-n} J(\eta^{-1}x)$, $x \in \mathbb{R}^n$, $\eta > 0$. Then, for any $u \in L$ (recall that, by **(P5)**, u is locally integrable), the function $J_\eta * u$ defined by

$$(J_\eta * u)(x) = \int_{\mathbb{R}^n} J_\eta(x - y)u(y)\,dy = \int_{\mathbb{R}^n} J(y)u(x - \eta y)\,dy$$

belongs to $C^\infty(\mathbb{R}^n)$. In particular, its restriction to G belongs to $C(G)$.

By Lemma 4.2 and properties of J,

$$\begin{aligned}
\|J_\eta * u - u\|_L &= \left\| \int_{\mathbb{R}^n} J(y)u(\cdot - \eta y)\,dy - \int_{\mathbb{R}^n} J(y)u(\cdot)\,dy \right\|_L \\
&= \left\| \int_{\mathbb{R}^n} \chi_{B(0,1)}(y)J(y)(u(\cdot - \eta y) - u(\cdot))\,dy \right\|_L \\
&\leq \int_{\mathbb{R}^n} \chi_{B(0,1)}(y)J(y)\|u(\cdot - \eta y) - u(\cdot)\|_L\,dy \\
&\leq \sup_{y \in B(0,1)} \|u(\cdot - \eta y) - u(\cdot)\|_L \\
&= \sup_{|h| < \eta} \|u(\cdot - h) - u(\cdot)\|_L.
\end{aligned}$$

Thus, according to condition (iii), there is $\delta > 0$ such that

$$\|J_\delta * u - u\|_L < \varepsilon/3. \tag{25}$$

[3] Note that a function $g \in L(\mathbb{R}^n)$ is said to be *L-mean continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|g(\cdot + h) - g(\cdot)\|_L < \varepsilon$ provided that $h \in \mathbb{R}^n$, $|h| < \delta$.

Now we are going to show that the set $\{(J_\delta * u)|_G : u \in K\}$ satisfies the conditions of the Arzelà-Ascoli Theorem in $C(G)$.

First, denoting by G_δ the neighbourhood of radius δ of G and using conditions **(P5)** and (i), we arrive at

$$\begin{aligned}
|(J_\delta * u)|_G(x)| &\leq \int_{\mathbb{R}^n} J(y)|u(x-\delta y)|\,dy \\
&\leq \Big(\sup_{y\in\mathbb{R}^n} J(y)\Big)\,\delta^{-n}\int_{B(x,\delta)}|u(z)|\,dz \\
&\lesssim \delta^{-n}\int_{G_\delta}|u(y)|\,dy \\
&\lesssim 1 \quad \text{for all } x\in G \text{ and } u\in K.
\end{aligned}$$

Second,

$$\begin{aligned}
&|(J_\delta * u)|_G(x+h) - (J_\delta * u)|_G(x)| \\
&\quad\leq \int_{\mathbb{R}^n} J(y)|u(x+h-\delta y) - u(x-\delta y)|\,dy \\
&\quad\leq \Big(\sup_{y\in\mathbb{R}^n} J(y)\Big)\,\delta^{-n}\int_{B(x,\delta)}|u(z+h)-u(z)|\,dz \\
&\quad\lesssim \delta^{-n}\int_{G_\delta}|u(z+h)-u(z)|\,dz \\
&\quad\lesssim \|u(\cdot+h)-u(\cdot)\|_L
\end{aligned}$$

for all $x, x+h \in G$ and all $u \in L$. Thus, by condition (iii), given any $\varepsilon_1 > 0$ there exists $\delta_1 > 0$ such that

$$|(J_\delta * u)|_G(x+h) - (J_\delta * u)|_G(x)| < \varepsilon_1$$

for all $u \in K$ and $x, x+h \in G$ with $|h| < \delta_1$.

By the Arzelà-Ascoli Theorem, the set $\{(J_\delta * u)|_G : u \in K\}$ is precompact in $C(G)$. Therefore, there exists a finite set $\{\psi_1, \ldots, \psi_m\}$ of functions in $C(G)$ with the following property: given any $u \in K$, there exists $j \in \{1, \ldots, m\}$ such that

$$|(J_\delta * u)(x) - \psi_j| < \frac{\varepsilon}{3\|\chi_G\|_L} \quad \text{for all } x \in G. \tag{26}$$

Thus, denoting by $\tilde{\psi}_j$ the extension of ψ_j by zero outside G, we obtain from (24), (25) and (26) that

$$\begin{aligned}
\|u-\tilde{\psi}_j\|_L &\leq \|(u-\tilde{\psi}_j)\chi_{\mathbb{R}^n\setminus G}\|_L + \|(u-\tilde{\psi}_j)\chi_G\|_L \\
&< \frac{\varepsilon}{3} + \|(u-J_\delta * u)\chi_G\|_L + \|(J_\delta * u - \tilde{\psi}_j)\chi_G\|_L \\
&< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

that is,

$$K \subset \bigcup_{j=1}^{m} B_\varepsilon(\tilde{\psi}_j),$$

which finishes the proof of the precompactness of K in L. $\square$

Remark 4.4. Part (a) of Theorem 4.3 remains true if the space L is replaced by any q-BFS $L(\mathbb{R}^n)$ coinciding with an r.i. BFS. Similarly, Part (b) of Theorem 4.3 remains true if the space L is replaced by any q-BFS $L(\mathbb{R}^n)$ coinciding with a BFS.

Now we would like to extend Theorem 4.3 to a q-BFS more general than the one considered in Remark 4.4. To this end, start by observing that, given a totally σ-finite measure space (R,μ), a function quasi-norm ρ and a positive number b, *the function* σ *defined by*

$$\sigma(f) := \big(\rho(f^{1/b})\big)^b, \quad f \in \mathcal{M}^+(R,\mu), \tag{27}$$

is also a function quasi-norm and the space

$$X(\sigma) := \{f \in \mathcal{M}(R,\mu) : \sigma(|f|) < \infty\}$$

is, according to Definition 3.2, the corresponding q-BFS. Since

$$\begin{aligned}\{f \in \mathcal{M}(R,\mu) : \sigma(|f|) < \infty\} &= \{f \in \mathcal{M}(R,\mu) : \rho(|f|^{1/b}) < \infty\} \\ &= \{f \in \mathcal{M}(R,\mu) : |f| = g^b \text{ for some } g \in X(\rho)\},\end{aligned}$$

we shall denote $X(\sigma)$ *also by* $X(\rho)^b$, *or simply by* X^b if it is clear that X refers to $X(\rho)$.

Definition 4.5. Given (R,μ) and a function quasi-norm ρ, the space $X(\rho)$ is called a *power q-BFS* if there exists $b \in (0,1]$ such that the space $X(\sigma)$, with σ from (27), coincides with a BFS.

Of course, any BFS is a power q-BFS (take $b = 1$ in (27)). However, more interesting is to note that a Lebesgue space $L_p(\mathbb{R}^n)$ with $0 < p < 1$ is also a power q-BFS. Indeed, choosing $b = p$, we obtain $\|f\|_{X(\sigma)} = \int_{\mathbb{R}^n} |f|$, and thus the space $X(\sigma)$ is the BFS $L_1(\mathbb{R}^n)$ (which means that $X(\rho) := L_p(\mathbb{R}^n)$ is a power q-BFS). Another, less trivial, example is given by a Lorentz space $L_{p,q}(\mathbb{R}^n)$ with $0 < p \le 1$ or $0 < q \le 1$. In this case we have, for each fixed positive b,

$$\begin{aligned}\|f\|_{X(\sigma)} &= \||f|^{1/b}\|^b_{X(\rho)} \\ &= \Big(\int_0^\infty (t^{1/p} f^*(t)^{1/b})^q \, \frac{dt}{t}\Big)^{b/q} \\ &= \Big(\int_0^\infty (t^{b/p} f^*(t))^{q/b} \, \frac{dt}{t}\Big)^{b/q}.\end{aligned} \tag{28}$$

Therefore, $X(\sigma)$ is $L_{p/b,q/b}(\mathbb{R}^n)$, depending on the choice of b. Since this coincides with a BFS if $p/b, q/b > 1$ (cf. [3, Chap. 4, Thm. 4.6]), the given Lorentz space $L_{p,q}(\mathbb{R}^n)$ is a power q-BFS (one takes $b \in (0, \min\{p,q\})$).

The notion of a power q-BFS is closely related to the so-called lattice convexity. We recall the latter in the following definition.

Definition 4.6. (see, e.g., [18, Def. 1.d.3] or [12]) If $X = X(\rho)$ is a q-BFL and $b \in (0, \infty)$, then X is said to be *b-convex* if for some $C \geq 0$ and any $g_1, \dots, g_m \in X$,

$$\left\| \left(\sum_{i=1}^{m} |g_i|^b \right)^{1/b} \right\|_X \leq C \left(\sum_{i=1}^{m} \|g_i\|_X^b \right)^{1/b}.$$

We have the following result.

Proposition 4.7. *Let $X = X(\rho)$ be a q-BFL and $b \in (0, \infty)$. Then the functional $f \mapsto \||f|^{1/b}\|_X^b$ is equivalent to a norm in X^b if and only if X is b-convex. In particular, given a q-BFS X, then X is a power q-BFS if and only if there is $b \in (0, 1]$ such that X is a b-convex q-BFL.*

Proof. Assuming that the functional $f \mapsto \||f|^{1/b}\|_X^b$ is equivalent to a norm in X^b, the b-convexity of X follows immediately.

On the contrary, assume now that X is a b-convex q-BFL and define

$$|||f||| := \inf \left(\sum_{i=1}^{m} \||f_i|^{1/b}\|_X^b \right) \qquad \text{for any } f \in X^b,$$

where the infimum is taken over all possible decompositions $f = \sum_{i=1}^{m} f_i$ with $f_i \in X^b$ and $m \in \mathbb{N}$. Clearly, $|||f||| \leq \||f|^{1/b}\|_X^b$ for all $f \in X^b$. As for the reverse inequality, given any $f \in X^b$ and any $\varepsilon > 0$, there exists a decomposition $f = \sum_{i=1}^{m} f_i$ such that

$$\sum_{i=1}^{m} \||f_i|^{1/b}\|_X^b \leq |||f||| + \varepsilon.$$

Therefore, using the b-convexity of X,

$$\||f|^{1/b}\|_X^b \leq \left\| \left(\sum_{i=1}^{m} |f_i| \right)^{1/b} \right\|_X^b \leq C^b \left(\sum_{i=1}^{m} \||f_i|^{1/b}\|_X^b \right) \leq C^b |||f||| + C^b \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\||f|^{1/b}\|_X^b \leq C^b |||f|||$.

So we have proved that the two expressions $\||f|^{1/b}\|_X^b$ and $|||f|||$ are equivalent. We leave to the reader to verify that the functional $||| \cdot |||$ is a norm in X^b. □

Remark 4.8. Note that criteria for b-convexity of some function lattices X have been studied – see, e.g., [14], [13] for the case when X is one of the Lorentz spaces $\Lambda^q(\omega)$ or $\Gamma^q(\omega)$, $0 < q < \infty$.

Theorem 4.9. *Let $L = L(\mathbb{R}^n)$ be a power q-BFS.*
(a) *If L coincides with a rearrangement invariant q-BFS and $K \subset \overline{C_0(\mathbb{R}^n)}$ is precompact in L, then K satisfies conditions* (i)-(iii) *of* Theorem 4.3.
(b) *Conversely, if $K \subset L_a$ satisfies conditions* (i)-(iii) *of* Theorem 4.3*, then K is precompact in L.*

Proof. With the hypotheses assumed, the proof of the necessity of conditions (i)-(iii) of Theorem 4.3 essentially follows the corresponding part of the proof of that theorem, with slight modifications. Therefore, we prove only that these conditions are sufficient for the precompactness of $K \subset L_a$ in L when L is a power q-BFS. Since the result follows immediately from Remark 4.4 if $b = 1$ in Definition 4.5, suppose that $b \in (0, 1)$.

We shall assume first that K contains only real functions. Denote

$$K_+^b := \{f_+^b : f \in K\}, \tag{29}$$

where the symbol $+$ in subscript indicates positive parts of functions and sets. It is easy to see that K_+^b is a subset of L^b, the latter coinciding, by Definition 4.5, with a BFS. We shall show that conditions (i)-(iii) of Theorem 4.3 hold for K_+^b and L^b instead of K and L, respectively. Therefore, by Remark 4.4, we conclude that K_+^b is precompact in L^b. Indeed, condition (i) for K_+^b and L^b follows from condition (i) for K and L, since

$$\|f_+^b\|_{L^b} := \|(f_+^b)^{1/b}\|_L^b = \|f_+\|_L^b \leq \|f\|_L^b.$$

Similarly, condition (ii) for K_+^b and L^b follows from condition (ii) for K and L, due to the inequality

$$\|\left(f_+^b\right)\chi_{\mathbb{R}^n \setminus G}\|_{L^b} \leq \|f\chi_{\mathbb{R}^n \setminus G}\|_L^b. \tag{30}$$

Finally, condition (iii) for K_+^b and L^b follows from the corresponding condition for K and L and the estimate

$$\begin{aligned} \|f_+^b(\cdot + h) - f_+^b(\cdot)\|_{L^b} &\leq \||f_+(\cdot + h) - f_+(\cdot)|^b\|_{L^b} \\ &= \|f_+(\cdot + h) - f_+(\cdot)\|_L^b \\ &\leq \|f(\cdot + h) - f(\cdot)\|_L^b; \end{aligned}$$

the first inequality is a consequence of the fact that $b \in (0, 1)$. Hence, as mentioned above, K_+^b is precompact in L^b.

Similarly, we can conclude that K_-^b is precompact in L^b, where

$$K_-^b := \{f_-^b : f \in K\}, \tag{31}$$

the symbol $-$ in subscript indicating negative parts of functions and sets.

Now take any sequence $\{f_k\}_k \subset K$ and consider the sequence

$$\{(f_k)_+^b\}_k \subset K_+^b. \tag{32}$$

Due to the precompactness of K_+^b in L^b, there is a subsequence $\{(f_{\tau_1(k)})_+^b\}_k$ converging in L^b, say to g. Consider an increasing sequence $\{E_m\}_{m \in \mathbb{N}}$ of compact subsets of $\mathbb{R}^n$ such that $\cup_{m \in \mathbb{N}} E_m = \mathbb{R}^n$. Since L^b coincides with a BFS, denoting by λ the Lebesgue measure in $\mathbb{R}^n$ and using **(P5)**, we obtain that

$$\int_{E_m} |(f_{\tau_1(k)})_+^b - g| \, d\lambda \underset{k}{\longrightarrow} 0 \quad \text{for any } m \in \mathbb{N}. \tag{33}$$

Consequently, there are subsequences $\{(f_{(\tau_2\circ\tau_1)(k)})_+^b\}_k$, $\{(f_{(\tau_3\circ\tau_2\circ\tau_1)(k)})_+^b\}_k$, ... converging λ-a.e. to g, respectively in E_2, E_3, ... Then the diagonal sequence

$$(f_{(\tau_2\circ\tau_1)(2)})_+^b,\ (f_{(\tau_3\circ\tau_2\circ\tau_1)(3)})_+^b,\ (f_{(\tau_4\circ\tau_3\circ\tau_2\circ\tau_1)(4)})_+^b,\ \dots$$

(which is a subsequence of $\{(f_k)_+^b\}_k$) converges λ-a.e. to g. Denote it simply by $\{(f_{\tau(k)})_+^b\}_k$. Consequently,

$$(f_{\tau(k)})_+ \underset{k}{\longrightarrow} |g|^{1/b} \quad \lambda\text{-a.e.}. \tag{34}$$

We recall that $g \in L^b$ therefore $|g|^{1/b} \in L$. Now, repeating the procedure above, but starting with $\{(f_{\tau(k)})_-^b\}_k \subset K_-^b$ instead of (32), we arrive at

$$(f_{\sigma(k)})_- \underset{k}{\longrightarrow} |h|^{1/b} \quad \lambda\text{-a.e.}, \quad \text{where } h \in L^b. \tag{35}$$

Conclusions (34) and (35) imply

$$f_{\sigma(k)} \underset{k}{\longrightarrow} |g|^{1/b} - |h|^{1/b} \quad \lambda\text{-a.e.}.$$

According to what was mentioned above, $|g|^{1/b} - |h|^{1/b} \in L$. Since convergence λ-a.e. yields local convergence in measure, we have just proved that

$$K \quad \text{is locally precompact in measure.} \tag{36}$$

The hypothesis $K \subset L_a$ implies that both K_+^b and K_-^b (given, respectively, by (29) and (31)) are subsets of $(L^b)_a$. Together with the conclusion that K_+^b and K_-^b are precompact in L^b and Theorem 3.17, this yields $K_+^b, K_-^b \subset \mathrm{UAC}(L^b)$. Since also

$$\begin{aligned} \|f\chi_{E_m}\|_L &= \|(f_+)\chi_{E_m} - (f_-)\chi_{E_m}\|_L \\ &\lesssim \|(f_+)\chi_{E_m}\|_L + \|(f_-)\chi_{E_m}\|_L \\ &= \|(f_+^b)\chi_{E_m}\|_{L^b}^{1/b} + \|(f_-^b)\chi_{E_m}\|_{L^b}^{1/b} \end{aligned}$$

for any $f \in K$ and any sequence $\{E_m\}_m$ of λ-measurable subsets of $\mathbb{R}^n$ with $E_m \underset{m}{\longrightarrow} \emptyset$ λ-a.e., we see that $K \subset \mathrm{UAC}(L)$. This fact, (36) and Theorem 3.17 imply that K is precompact in L.

In the case in which K contains complex functions, we get the result applying the above method successively to the real parts and then to the imaginary parts of functions. □

Remark 4.10. In part (a) of Theorem 4.9 one can use the assumptions that L^b coincides with an r.i. BFS, $L = L_a$ and $\overline{K \subset L}$ instead of the hypotheses that L coincides with an r.i. q-BFS and $K \subset \overline{C_0(\mathbb{R}^n)}$. Indeed, from the assumptions made now one can prove that $\overline{C_0(\mathbb{R}^n)} = L \supset K$, with arguments similar to the ones used above (replacing Theorem 3.17 by Lemma 3.15), and that L coincides with an r.i. q-BFS.

5 Abstract Besov spaces and compact embeddings

In what follows, $\mathbb{R}^n$ and $(0,1)$ are endowed with the corresponding Lebesgue measures. The *modulus of continuity* of a function f in a q-BFS $L = L(\mathbb{R}^n)$ over $\mathbb{R}^n$ is given by

$$\omega_L(f,t) := \sup_{\substack{h \in \mathbb{R}^n \\ |h| \le t}} \|\Delta_h f\|_L, \quad t > 0.$$

The following is an extension of the definition of generalized Besov spaces considered in [10] to the setting of quasi-Banach function spaces.

Definition 5.1. Let $L = L(\mathbb{R}^n)$ be a q-BFS and let Y be a q-BFL over $(0,1)$ satisfying

$$\|1\|_Y = \|\chi_{(0,1)}\|_Y = \infty. \tag{37}$$

The abstract Besov space $B(L,Y)$ is the set of all $f \in L$ such that the quasi-norm

$$\|f\|_{B(L,Y)} := \|f\|_L + \|\omega_L(f,\cdot)\|_Y$$

is finite.

Remark 5.2. The assumption $\|1\|_Y = \infty$ in the Definition 5.1 is natural (otherwise $B(L,Y) = L$, which is not of interest). Also notice that assumption (37) violates axiom **(P4)**. Consequently, Y is not a q-BFS.

In what follows we consider the space

$$L(\Omega) := \{f|_\Omega : f \in L\},$$

for a measurable set $\Omega \subset \mathbb{R}^n$ with $|\Omega|_n > 0$, quasi-normed by

$$\|f|_\Omega\|_{L(\Omega)} := \|f\chi_\Omega\|_L.$$

Theorem 5.3. *Let $L = L(\mathbb{R}^n)$ be a power q-BFS. In the case that L does not coincide with a BFS, we assume that $L = L_a$. Let $B(L,Y)$ be an abstract Besov space and let Ω be a bounded measurable subset of $\mathbb{R}^n$ with $|\Omega|_n > 0$. Then*

$$B(L,Y) \hookrightarrow\hookrightarrow L(\Omega)$$

(this means that the restriction operator $f \mapsto f|_\Omega$ is compact from $B(L,Y)$ into $L(\Omega)$).

Proof. We are going to prove that

$$\{f|_\Omega : f \in B(L,Y), \|f\|_{B(L,Y)} \le 1\}$$

is precompact in $L(\Omega)$.

Since Ω is bounded, there is $R_0 \in (0,\infty)$ such that $\overline{\Omega} \subset B(0,R_0)$. Take a Lipschitz continuous function φ on $\mathbb{R}^n$ satisfying $0 \le \varphi \le 1$, $\varphi = 1$ on Ω and $\varphi = 0$ on $\mathbb{R}^n \setminus B(0,R_0+1)$. Then, for all $f \in L$,

$$\|f|_\Omega\|_{L(\Omega)} = \|f\chi_\Omega\|_L = \|\varphi f \chi_\Omega\|_L \le \|\varphi f\|_L.$$

Therefore, it is sufficient to prove that the set

$$K := \{\varphi f : f \in B(L,Y), \|f\|_{B(L,Y)} \le 1\}$$

is precompact in L.

By Theorems 4.3, 4.9 and Remark 4.4, it is enough to verify that conditions (i)-(iii) of Theorem 4.3 hold for the K and L considered here.

(i) Let $\varphi f \in K$, with $\|f\|_{B(L,Y)} \le 1$. Since $\|\varphi f\|_L \le \|f\|_L \le \|f\|_{B(L,Y)}$, the set K is bounded in L.

(ii) The set $G := B(0,R_0+1)$ is compact in $\mathbb{R}^n$. If $\varphi f \in K$, then $\|\varphi f \chi_{\mathbb{R}^n \setminus G}\|_L = 0 < \varepsilon$ for all $\varepsilon > 0$.

(iii) Given $f \in L$ and $x, h \in \mathbb{R}^n$,

$$\begin{aligned} |\Delta_h(\varphi f)(x)| &\le \|\varphi\|_{\infty,\mathbb{R}^n} |\Delta_h f(x)| + \|\Delta_h \varphi\|_{\infty,\mathbb{R}^n} |f(x)| \\ &\lesssim |\Delta_h f(x)| + |h||f(x)|. \end{aligned} \tag{38}$$

Hence,

$$\|\Delta_h(\varphi f)\|_L \lesssim \|\Delta_h f\|_L + |h|\|f\|_L.$$

If $\varphi f \in K$, with $\|f\|_{B(L,Y)} \le 1$, then conditions **(P2)**, **(P1)**, together with the facts $Y \subset \mathcal{M}_0(0,1)$ (recall Lemma 3.3) and that $\omega_L(f,\cdot)$ is non-decreasing, imply that

$$1 \ge \|f\|_{B(L,Y)} \ge \|\omega_L(f,\cdot)\chi_{(T,1)}\|_Y \ge \omega_L(f,T)\|\chi_{(T,1)}\|_Y \quad \text{for } T \in (0,1). \tag{39}$$

Moreover, **(P3)** and (37) yield that $\lim_{T\to 0+} \|\chi_{(T,1)}\|_Y = \infty$. Consequently, we obtain from (39) that

$$\lim_{T\to 0+} \omega_L(f,T) = 0 \quad \text{uniformly with respect to } f \text{ such that } \varphi f \in K.$$

Combining with (38) and recalling that $\|f\|_L \le \|f\|_{B(L,Y)}$, we get that

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall (\varphi f) \in K,\ |h| < \delta \Rightarrow \|\Delta_h(\varphi f)\|_L < \varepsilon.$$

Applying Theorem 4.3 and Remark 4.4, or Theorem 4.9, we obtain the precompactness of K in L and the proof is complete. □

Corollary 5.4. *Let $L = L(\mathbb{R}^n)$ be a power q-BFS. In the case that L does not coincide with a BFS, assume that $L = L_a$. Let $B(L,Y)$ be an abstract Besov space. Let Ω be a bounded measurable subset of $\mathbb{R}^n$ with $|\Omega|_n > 0$ and let $Z(\Omega)$ be a q-BFS over Ω (with the restricted Lebesgue measure). Assume that the restriction operator maps $B(L,Y)$ into $(Z(\Omega))_a$. Then this operator is compact if and only if*

$$K := \{f|_\Omega : f \in B(L,Y), \|f\|_{B(L,Y)} \le 1\} \subset UAC(Z(\Omega)). \tag{40}$$

Proof. As a consequence of Theorem 5.3, the set K in (40) is precompact in $L(\Omega)$. Therefore, by Theorem 3.17 and Remark 3.18, such a K is locally precompact in measure in $\mathcal{M}_0(\Omega)$. Hence, by Corollary 3.19, the set K is precompact in $Z(\Omega)$ if and only if $K \subset \mathrm{UAC}(Z(\Omega))$. □

Remark 5.5. The assumptions for the "if" part of the Corollary 5.4 can be relaxed. In fact, from Definition 3.12 and Theorem 3.17, we have that if (40) holds, then the restriction operator takes $B(L, Y)$ compactly into $Z(\Omega)$. That is, we don't need to assume *a priori* that the restrictions of the elements of $B(L, Y)$ are in $(Z(\Omega))_a$ (since this is a consequence of (40), by the homogeneity of the quasi-norm). Actually, here we don't even have to assume *a priori* that those restrictions belong to $Z(\Omega)$ (again, this is a consequence of (40)).

6 Applications

We are going to apply Corollary 5.4 and Remark 5.5 to the case when $Z(\Omega)$ is the Lorentz-type space $L_{p,q;w}(\Omega)$ introduced in Section 2. We start with the following criterion for a subset of measurable functions to have uniformly absolutely continuous quasi-norm in $L_{p,q;w}(\Omega)$.

Proposition 6.1. *Let $0 < p, q \leq \infty$, let Ω be a measurable subset of $\mathbb{R}^n$ with $0 < |\Omega|_n < \infty$ and $w \in \mathcal{W}(0, |\Omega|_n)$. Assume that* (1) *and* (5) *hold. If $K \subset \mathcal{M}(\Omega)$ is such that*

$$\lim_{\delta \to 0+} \sup_{u \in K} \|t^{1/p-1/q} w(t) u^*(t)\|_{q;(0,\delta)} = 0, \tag{41}$$

then $K \subset UAC(L_{p,q;w}(\Omega))$.

Proof. Given any $\delta \in (0, |\Omega|_n)$ and $u \in K$,

$$\begin{aligned} &\|t^{1/p-1/q} w(t) u^*(t)\|_{q;(0,|\Omega|_n)} \\ &\quad \lesssim \|t^{1/p-1/q} w(t) u^*(t)\|_{q;(0,\delta)} + u^*(\delta) \|t^{1/p-1/q} w(t)\|_{q;(\delta,|\Omega|_n)} \\ &\quad \leq \|t^{1/p-1/q} w(t) u^*(t)\|_{q;(0,\delta)} + u^*(\delta) B_{p,q;w}(|\Omega|_n) =: V(\delta, u). \end{aligned}$$

Combining with (41) and (1), one can choose $\delta \in (0, |\Omega|_n)$ such that $V(\delta, u)$ is finite. Therefore, $K \subset L_{p,q;w}(\Omega)$. Moreover, given any $\varepsilon > 0$, property (41) implies that there is $\delta > 0$ such that for all $u \in K$ and all measurable E, with $|E|_n < \delta$,

$$\|u\chi_E\|_{p,q;w;\Omega} \leq \|t^{1/p-1/q} w(t) u^*(t) \chi_{[0,\delta)}(t)\|_{q;(0,|\Omega|_n)} < \varepsilon,$$

which proves that $K \subset \mathrm{UAC}(L_{p,q;w}(\Omega))$, due to Remark 3.14(i). □

Let $0 < s < 1$ and $0 < p, q \leq \infty$. Choosing $L = L_p(\mathbb{R}^n)$ and $Y = \{g \in \mathcal{M}(0, 1) : \|g\|_Y = \|t^{-s-1/q} g(t)\|_{q;(0,1)} < \infty\}$ in Definition 5.1, we obtain the well-known Besov spaces (of small smoothness s, defined by differences), equipped with the quasi-norm

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} := \|f\|_p + \|t^{-s-1/q} \omega_{L_p(\mathbb{R}^n)}(f, t)\|_{q;(0,1)}.$$

Recently the so-called growth envelope functions for such spaces have been obtained in [11, Prop. 2.5, Thm. 2.7]. We recall the result.

Proposition 6.2. *Let $0 < s < 1$, $0 < p, q \le \infty$.*

(i) *If $s < \frac{n}{p}$, then*

$$\sup_{\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \le 1} f^*(t) \approx t^{-1/p+s/n} \quad \textit{as } t \to 0+ \,.$$

(ii) *If $s = \frac{n}{p}$ and $q > 1$, then*

$$\sup_{\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \le 1} f^*(t) \approx |\log t|^{1/q'} \quad \textit{as } t \to 0+ \,.$$

(iii) *In all the remaining cases,*

$$\sup_{\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \le 1} f^*(t) \approx 1 \quad \textit{as } t \to 0+ \,.$$

Using Proposition 6.2 and our results, we obtain compact embeddings of these Besov spaces.

Theorem 6.3. *Let $0 < s < 1$ and $0 < p, q, r, u \le \infty$. Let Ω be a bounded measurable subset of $\mathbb{R}^n$, $|\Omega|_n > 0$, and $w \in \mathcal{W}(0, |\Omega|_n)$. Assume that* (1), (5) *and one of the following conditions are satisfied:*

(i) *$s < \frac{n}{p}$ and $\lim_{\delta \to 0+} \|t^{1/r-1/u-1/p+s/n} w(t)\|_{u;(0,\delta)} = 0$;*

(ii) *$s = \frac{n}{p}$, $q > 1$ and $\lim_{\delta \to 0+} \|t^{1/r-1/u} |\log t|^{1/q'} w(t)\|_{u;(0,\delta)} = 0$;*

(iii) *$s > \frac{n}{p}$, or $s = \frac{n}{p}$ and $0 < q \le 1$, and $\lim_{\delta \to 0+} \|t^{1/r-1/u} w(t)\|_{u;(0,\delta)} = 0$.*

Then

$$B^s_{p,q}(\mathbb{R}^n) \hookrightarrow\hookrightarrow L_{r,u;w}(\Omega).$$

Proof. Observing that

$$\begin{aligned} 0 \le & \sup_{\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \le 1} \|t^{1/r-1/u} w(t) (f|_\Omega)^*(t)\|_{u,(0,\delta)} \\ \le & \ \|t^{1/r-1/u} w(t) \sup_{\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \le 1} f^*(t)\|_{u;(0,\delta)}, \end{aligned}$$

the conclusion follows from Propositions 6.2, 6.1, Corollary 5.4 and Remark 5.5. □

Remark 6.4. When more precise estimates than the growth envelope functions are known, our approach may lead to weaker sufficient conditions for compactness. For example, when $s < n/p$ (with $0 < s < 1$) and $0 < q \le u \le \infty$, we have, for small $\delta > 0$, from [11, Cor. 3.3] that

$$\|t^{1/p-s/n-1/u} f^*(t)\|_{u;(0,\delta)} \lesssim \|f\|_{B^s_{p,q}(\mathbb{R}^n)} \quad \text{for all } f \in B^s_{p,q}(\mathbb{R}^n).$$

Consequently,

$$\sup_{\|f\|_{B^s_{p,q}(\mathbb{R}^n)}\le 1} \|t^{1/p-s/n-1/u} f^*(t)\|_{u;(0,\delta)} \lesssim 1 \quad \text{for all small } \delta > 0.$$

Therefore, with $\frac{1}{r} := \frac{1}{p} - \frac{s}{n}$,

$$\begin{aligned} 0 \quad &\le \sup_{\|f\|_{B^s_{p,q}(\mathbb{R}^n)}\le 1} \|t^{1/r-1/u} w(t) (f|_\Omega)^*(t)\|_{u;(0,\delta)} \\ &\lesssim \operatorname{ess\,sup}_{t\in(0,\delta)} w(t). \end{aligned}$$

Hence, the assumption $\lim_{t\to 0+} w(t) = 0$ (for the mentioned conjugation of parameters) guarantees that $B^s_{p,q}(\mathbb{R}^n) \hookrightarrow\hookrightarrow L_{r,u;w}(\Omega)$.

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